BH EXCISING TECHNIQUES

- BLACK HOLE (EVENT HORIZON) ⇒ TRAPPED SURFACE
  ⇒ PHYSICAL SINGULARITY

- BH (EH) ⇒ TS ⇒ SING (PENROSE / HAWKING / ELLIS MID 60'S + ⇒ SEE L.S.S. of S.T BY HAWKING & ELLIS)

  RIGOROUS THM^S CONCERNING GRAV. COLLAPSE BASED ON "GLOBAL TECHNIQUES" (GEOMETRICAL, COORDINATE IND., MAKE MINIMAL USE of PDEs) → DEMONSTRATE "GEODESIC INCOMPLETENESS" → MINIMAL QUANT. INFO CONCERNING SING.

- COLLAPSE CALCULATIONS (NR); LARGELY COMPLEMENTARY SUGGEST SAME BASIC PICTURE, BUT SINGULARITY "TYPICALLY" (I.E. IF WE CHOSE TO INTEGRATE TOWARDS SING) MANIFEST IN INVARIANTS (R_{abcd}) → ∞ etc.

- CLEARLY, NEED TO AVOID PHYSICAL SINGULARITY IN SIMULATION (≡ NUMERICAL ANALYSIS of PDEs), OR BE PREPARED TO INVEST HEAVILY IN TREATMENT of SING (WON'T CONSIDER 2ND OPTION – GOOD RESEARCH PROBLEM)

- TRADITIONAL NR APPROACH

  - ASSUME PHYS SING "$r=0$" MUST BE AVOIDED
  - STRATEGY

    S.1) AVOID AT $t=0$

    S.2) CONTINUE TO AVOID AT $t = t + \Delta t$

• TWIST : TWO BASIC TYPES OF BH COMPUTATION

(A) BH FORMS AT $t = t_{BH}$

(B) BH EXISTS AT $t = 0$

CASE A) BH-FORMING CALCULATION (IMKG, PA, etc)

• $\Sigma(0) \sim \Sigma_{FLAT}(0)$, SPATIAL COORDS. x^i COVER $\mathbb{R}^3$ (INCL "$r=0$") ⟹ SINGULARITY AVOIDED AT $t=0$

• USE SLICING CHOICE TO "AVOID" SINGULARITY (FUNDAMENTALLY, MAKE USE OF FACT THAT PAST(ST − BH) CONTAINS NONE of BH − I.E. CAN CHOOSE TO EVOLVE SOME, ALL, NONE OF BH INTERIOR)

EXAMPLES: (SPH SYMM, BUT CONSIDERATIONS EXTEND TO 2- 3-d)

POLAR-SLICING / AREAL SPATIAL COORDINATES : WHEN BH FORMATION IMMINENT, THEN

$$\alpha(r,t) \to 0 \qquad r \leq R_{BH}$$

→ COLLAPSE of LAPSE; SLICES (ALWAYS CONNECTED TO $r=0$) "APPROACH" NULL LIMIT SURFACE ($r = 2M$)

MAXIMAL SLICING / ISOTROPIC x^i

MAXIMAL SLICING / AREAL x^i

→ SIMILAR BEHAVIOUR, $\alpha(r,t) \to 0$, $r \leq R_{LIM}$

BUT $R_{LIM} < R_{BH}$

- KEY PROBLEM: NATURE of COORDS + DOMAIN COVERED BY COORDS ⟹ COORDINATE SINGULARITY FORMS DURING EVOLUTION

- HEURISTIC PICTURE of COORD SING ; IMPACT ON SOL^N of EOM.

- FOCUS ON SLICING, t, (ANALYSIS OF SPATIAL COORDS x^i SHOULD //, SINCE BHS FUND. "NULL" OBJECTS) CAN THEN LITERALLY THINK OF THE OBSERVER FIELD WITH 4-VELOCITY $v^a = n^a$ "LAYING DOWN" THE TIME COORD

- INTUITIVELY, IF THESE OBSERVERS PENETRATE THE HORIZON (OR COME ARBITRARILY CLOSE), WILL HAVE, E.G.

$$a^a = n^b \nabla_b n^a \to \infty$$

AND A "PILE UP" OF THE SLICES, IN ORDER THAT THE PHYSICAL SINGULARITY BE AVOIDED.

- NOW CONSIDER (E.G.) EVOLUTION EQN FOR $K^a{}_b$

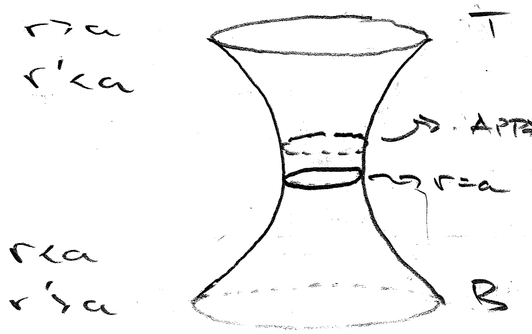
$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - \underbrace{D^a D_b \alpha}_{\hookrightarrow\, D^a(\alpha a_b)} + \alpha (R^a{}_b + \dots)$$

TROUBLE!!

- BASICALLY, FROM PDE POINT of VIEW, COORD SING. IS JUST AS PATHOLOGICAL AS PHYS SING.

CASE B) EXTANT BH CALCULATION (SMARR/NCSA/POTSDAM CALCULATIONS mid 70'S → PRESENT)

- USE SPECIAL GEOMETRIC PROPERTY TO EXCISE "r=0" FROM DOMAIN AT $t=0$, E.G. INTRODUCE "ISOMETRY" THROAT AT $r=a$ FOR SINGLE BH



- TOP AND BOTTOM "SHEETS" T : B ISOMETRIC W.R.T "INVERSION THROUGH SPHERE" $r \to a^2/r$

$$r' = \frac{a^2}{r}$$

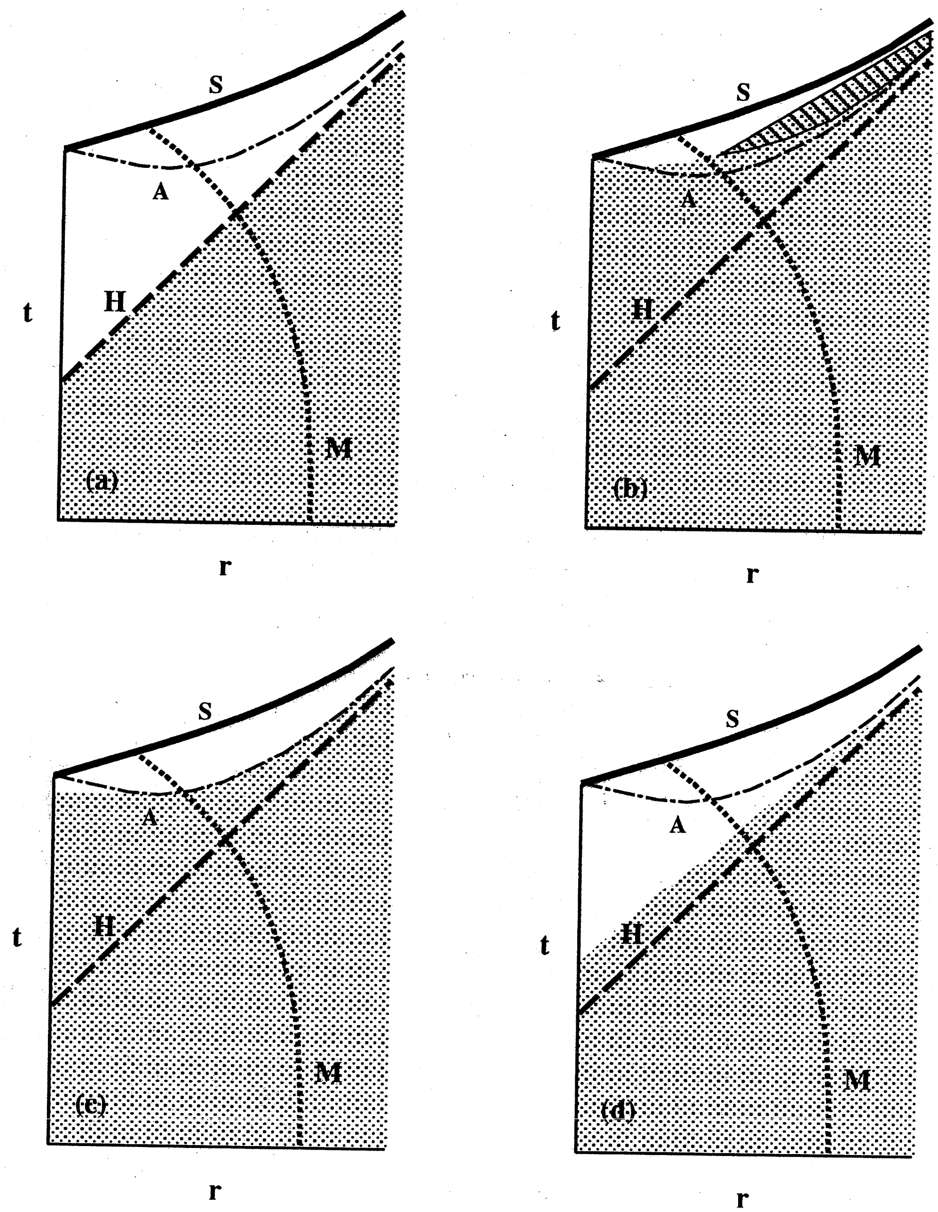
- t = CONSTANT SURFACES MUST PENETRATE AH (OR REP. OF INITIAL DATA WILL NOT BE REGULAR); MAINTAIN SYMMETRY COND. FOR $t > 0$

- COORD. SING MUST DEVELOP AS BEFORE (BY SAME HEURISTIC REASONING); CALCS. HAVE TYPICALLY USED MAXIMAL SLICING, LAPSE COLLAPSES INSIDE BH etc.

- THIS TRADITIONAL SINGULARITY-AVOIDING APPROACH LED TO SOMEWHAT ABSURD, BUT SUSTAINED EFFORT WHERE COMPUTATIONAL COST (I.E. CPU TIME), $C(t)$, TO SIMULATE BH SPACETIME ON $0 < t < T$ WENT LIKE

$$C(T) \sim \exp^{\sigma T} \qquad (*)$$

IMPORTANTLY, MUCH OF THIS EFFORT PURPORTED TO STUDY



Computational Domains for Black Hole Calculations. Schematic depiction of computational domains (shaded regions) employed in various approaches to the simulation of a black hole which forms from matter collapse. **M** is the outer surface of the infalling matter, **H** is the event horizon, **A** is the locus of outer apparent horizons and **S** is the spacetime singularity. (Adapted from Christodoulou, *Comm. Pure Appl. Math*, **44**, 339 (1991).)

(a) Ideal calculation. Computational domain contains *only* those events which are external to or coincident with event horizon **H**.

(b) Traditional numerical relativity calculation. Code avoids singularity, **S**, but uses coordinate freedom to "freeze" evolution near **S**, resulting in unbounded growth in evolved dynamical variables in diagonally shaded region.

(c) Apparent horizon tracking calculation. Code locates, then tracks, position of outer apparent horizon on constant time surfaces and can thus excise region lying interior to **A** from computational domain. Values of evolved dynamical variables do *not* grow without bound.

(d) Event horizon tracking calculation. Computational domain coincides more nearly with black hole exterior than in calculation (c), avoiding some potential problems with tracking apparent horizons. Efficient realization of such a scheme might involve a spacetime multigrid approach.

ASTROPHYSICS; INTERESTED IN GRAV. WAVE SIGNALS AS SEEN AT $r \to \infty$, I.E. IN BH EXTERIOR.

- (*) IS TO BE COMPARED WITH ANY "REASONABLE" ASTROPHYSICAL CALC. (E.G. SIMULATION of A STABLE STAR) WHERE

$$C(T) \sim T \qquad (**)$$

BH CALCS: 1978 2-BH CALC BLOWS UP AT $T \sim 10M$
1998 " " AT $T \sim 500M$

- NO REASON NOT TO DEMAND SCALING BEH. (**) FOR ASTROPHYSICALLY-MOTIVATED CALCS
  - NOTHING SPECIAL ~~HAPPENS AT~~ EH (PRINCIPLE of EQUIVALENCE, LOCAL FLATNESS ...)
  - CAN IGNORE WHAT GOES ON INSIDE BH

- PARADIGM SHIFT

TRADITIONAL BH CALC. $\Rightarrow$ EXCISING CALC.

1980s	1990s
"THE TALK"	"THE WALK"
UNRUH	THORNBURG
THORNBURG	SEIDEL & SUEN
CHOPTUIK	SCHEEL et al
	→ MARSA & CHOPTUIK
	ANNINOS et al
	COOK et al (CHUG)

B.H. EXCISING STRATEGY

S.1 DON'T EVOLVE INTERIOR of BH'S (INSIDES EH'S)

S.2 BECAUSE EH LOCATION CANNOT BE A DYNAMICAL VBL, USE APPROX. TO EH LOCATION BASED ON AH LOCATION, ASSUME THAT $R_{AH}(t) < R_{EH}(t)$

(COSMIC CENSORSHIP)

RECALL: AH = OUTERMOST MARGINALLY TRAPPED SURFACE

$$D^a s_a - K + s^a s^b K_{ab} = 0 \qquad \text{(AHEqn)}$$

DISCUSS DIAGRAM: "COMPUTATIONAL DOMAINS FOR BH CALCULATIONS"

NOTE: PARADIGM SHIFT FOCUSED ATTN ON TECHNIQUES/ALG. FOR SOLVING (AHEqn), STILL ACTIVE AREA of RESEARCH

- HOWEVER, PROBLEM ESSENTIALLY TRIVIAL IN 1-D WHICH WILL BE OUR FOCUS (ad hoc METHODS OK.)

- WILL NOW STUDY APPROACH DISCUSSED IN MARSA ; CHOPTUIK (PRD, 54, 4929-4943)

- EMKG WITH PRE-EXISTING BH ; EXCISING

(SCALAR FIELD - BH ACCRETION / SCATTERING, SPH. SYMM.)

- RECALL (HW 2.2) STATIC (SCHWARZSCHILD) SOL^N IN IEF COORDS

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}\,dt\,dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$$

COMPARISON WITH GENERAL FORM

$$ds^2 = (-\alpha(r)^2 + a^2(r)\beta^2(r))dt^2 + 2a^2\beta\,dt\,dr + a^2 dr^2 + r^2 d\Omega^2$$

$$G_6(r) = 1$$

$$\alpha = \left(\frac{r}{r+2m}\right)^{\frac{1}{2}} \qquad a = \alpha^{-1} = \left(\frac{r}{r+2m}\right)^{-\frac{1}{2}}$$

$$\beta = \frac{2m}{r+2m} \qquad \frac{\alpha}{a} = 1 - \beta$$

- ALL KINEMATICAL / DYNAMICAL VARS AND DERIVS. PERFECTLY WELL-BEHAVED AT $r = 2m$

- DEFINING IEF COORDS FROM 3+1 PERSPECTIVE
  (1) SPATIAL COORD., r, CHOSEN TO BE AREAL
  (2) TIME COORD, t, CHOSEN SO THAT

$$v^a = \left(\frac{\partial}{\partial t}^a - \frac{\partial}{\partial r}^a\right) \text{ IS NULL } \quad (v^a v_a = 0)$$

(RECALL: GEN. CHARACTERISTIC SPEEDS $c_\pm = -\beta \pm \frac{\alpha}{a}$, $c_- = -\beta - (1-\beta) = -1$ ✓)

- GENERALIZATION OF IEF TO TIME-DEPENDENT CASE

- TWO APPROACHES
  (A) MINIMAL MODIFICATION & "AH-LOCKING"
  (B) DIRECT IMPL. AS DEFINED ABOVE

- HISTORICALLY, (A) AND SIMILAR APPROACHES BY OTHER AUTHORS WAS INVEST. IN DETAIL FIRST, MOSTLY DUE TO (PROBABLY) MISGUIDED NOTION THAT TYING AH LOC^N TO CONSTANT SPATIAL COORD WAS "PARTICULARLY GOOD THING" → WE WILL FOCUS ON (B)

EMKGBH PROBLEM IN IEF COORDS

* ASSUME AT $t=0$, $\Sigma(0)$ DESCRIBES APPROX STATIC BH WITH MASS M_{BH}, WITH SOME NON-SINGULAR SCALAR FIELD DISTRIBUTION $\Phi(r,0)$, $\Pi(r,0)$ "SUPERIMPOSED"

GENERAL METRIC : $\alpha, a, \beta, b \; : \; (r,t)$

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta\, dt\, dr + a^2 dr^2 + r^2 b^2 d\Omega^2$$

COORDINATE CONDITIONS

(1) AREAL r :

$$\boxed{\begin{aligned} b(r,t) &= 1 \\ \dot b(r,t) &= 0 \end{aligned}} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

(2) INCOMING-NULL-BASED t: $\quad g_{\mu\nu}\left(\frac{\partial}{\partial t}^{\mu} - \frac{\partial}{\partial r}^{\mu}\right)\left(\frac{\partial}{\partial t}^{\nu} - \frac{\partial}{\partial r}^{\nu}\right) = 0$

$$\boxed{\alpha = a(1-\beta)} \qquad (3)$$

* WILL USE (3) TO ELIMINATE $\alpha(r,t)$ FROM E.O.M.

* SCHEMATIC SKETCH of EOM / COORD. COND.

$\begin{cases} \alpha & \text{- ALGEBRAICALLY FIXED} \\ \beta \end{cases}$

	EVOLUTION	CONSTRAINTS
a	$\dot a =$	$a' =$
b	$\dot b = 0$	
$K^r{}_r$	$\dot K^r{}_r =$	
$K^\theta{}_\theta$	$\dot K^\theta{}_\theta =$	$K^\theta{}_\theta{}' =$
Φ	$\dot \Phi =$	
Π	$\dot \Pi =$	

- NOTE: "FULLY CONSTRAINED" SCHEME NOT POSSIBLE HERE; WILL HAVE TO USE AT LEAST ONE of $\dot{K}^r{}_r, \dot{K}^\theta{}_\theta$

- COORDINATE CONDITIONS (CONT.)

(1) $\alpha = a(1-\beta)$

(2) $b=1, \quad \dot{b}=0$

(SS34) $\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$

$\rightarrow \beta = r\alpha K^\theta{}_\theta = ra(1-\beta)K^\theta{}_\theta$

$$\Rightarrow \quad \boxed{\begin{aligned} \beta &= \frac{raK^\theta{}_\theta}{1+raK^\theta{}_\theta} = \frac{H}{1+H} && (4) \\ H &= raK^\theta{}_\theta && (5) \end{aligned}}$$

- SCALAR FIELD EOM: STRESS-ENERGY COMP.

- DEFINE, AS USUAL

$$\boxed{\begin{aligned} \Phi &\equiv \phi' && (6) \\ \Pi &\equiv \frac{a}{\alpha}(\dot{\phi} - \beta\phi') && (7) \end{aligned}}$$

(SS37) $$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)'$$

$$\boxed{\dot{\Phi} = \left(\beta\Phi + (1-\beta)\Pi\right)'} \qquad (8)$$

(SS38) $$\boxed{\dot{\Pi} = \frac{1}{r^2}\left(r^2\left(\beta\Pi + (1-\beta)\Phi\right)\right)'} \qquad (9)$$

$$\rho = S^r{}_r = \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (10)$$

$$S^\theta{}_\theta = S^\phi{}_\phi = \frac{\Pi^2 - \Phi^2}{2a^2} \qquad (11)$$

- CONSTRAINT EQUATIONS

- HAMILTONIAN CONSTRAINT (SS18, SS31)

$$\boxed{\frac{a'}{a} + \frac{a^2-1}{2r} + \frac{a^2 r}{2} K^\theta{}_\theta\left(2K^r{}_r + K^\theta{}_\theta\right) - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0} \qquad (12)$$

- MOMENTUM CONSTRAINT (SS32)

$$\boxed{K^\theta{}_\theta{}' + \frac{\left(K^\theta{}_\theta - K^r{}_r\right)}{r} - 4\pi\frac{\Phi\Pi}{a} = 0} \qquad (13)$$

- GEOMETRIC EVOLUTION EQUATIONS: (AFTER SUBS $\alpha = a(1-\beta)$)

(SS33) $$\boxed{\dot{a} = -a^2(1-\beta)K^r{}_r + (a\beta)'} \qquad (14)$$

(SS35)
$$\boxed{\dot{K}^r{}_r = \beta K^r{}_r{}' + a(1-\beta)K^r{}_r\left(K^r{}_r + 2K^\theta{}_\theta\right) + \frac{(\beta-1)}{a}\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 - \frac{2}{r}\frac{a'}{a} + 8\pi\Phi^2\right] + \frac{\beta' a'}{a^2} + \frac{\beta''}{a}} \qquad (15)$$

(SS36)
$$\boxed{\dot{K}^\theta{}_\theta = \beta K^\theta{}_\theta{}' + a(1-\beta)K^\theta{}_\theta\left(K^r{}_r + 2K^\theta{}_\theta\right) + \frac{(1-\beta)}{r^2}\left(a - \frac{1}{a}\right) + \frac{\beta'}{ar}} \qquad (16)$$

LOCATING / TRACKING APPARENT HORIZONS (TS)

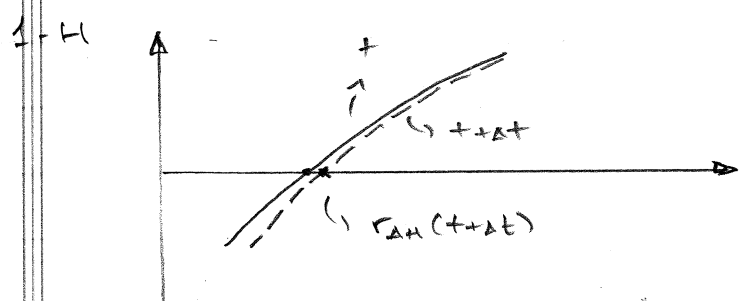
$$D_a s^a - K + s^a s^b K_{ab} = 0 \qquad s^a s_a = 1, \quad s^a \perp TS$$

(SS49)
$$(rb)' = arbK^\theta{}_\theta$$

• WITH THE IEF COORDINATE CONDITION, $b=1$, WE HAVE

$$\boxed{1 - arK^\theta{}_\theta = 1 - H = 0}$$

• AH LOCATION ⇒ DETECTING 0-CROSSING OF $1 - H = 1 - arK^\theta{}_\theta$



IDEA: CONTINUOUSLY TRACK $r_{AH}(t)$, USE AS INNER EDGE OF COMP DOMAIN.

APPLY EV. EQN'S ON $r_{AH}(t) \le r < r_{max}$; WILL GENERALLY NEED MODIFIED DIFF EQN'S ("ONE-SIDED") AT/NEAR $r = r_{AH}(t)$

CHECK: FOR SCHWARZSCHILD SOL'N WE HAVE

$$a = \frac{(r+2m)^{\frac{1}{2}}}{r^{\frac{1}{2}}}$$

$$K^0{}_0 = \frac{\beta}{ra(1-\beta)} = \frac{2m(r+2m)}{r^{3/2}(r+2m)^{3/2}}$$

$$\rightarrow \quad arK^0{}_0 = \frac{2m}{r}$$

$$1 - \frac{2m}{r} \qquad \begin{array}{ll} > 0 & r > 2m \\ = 0 & r = 2m \\ < 0 & r < 2m \end{array}$$

- CONSIDER INCOMING NULL VECTOR IN SCHWARZ. SPACETIME IN IEF COORDS

$$l^a l_a = 0 \Rightarrow l^a = [1, -1, 0, 0]$$

WE ~~HAVE~~

$$ds^2 = \left(-1 + \frac{2M}{r}\right)dt^2 + \frac{4M}{r}dt\,dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$$

$$l_\mu = g_{\mu\nu} l^\nu$$

$$l_t = (g_{tt} - g_{tr}) = -1$$

$$l_r = (g_{rt} - g_{rr}) = -1$$

AND ~~WE~~ CAN WRITE THE METRIC AS

$$\boxed{\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + H(r)\, l_\mu l_\nu \\ H(r) &= \frac{2M}{r} \end{aligned}} \qquad (**)$$

WHERE $\eta_{\mu\nu}$ IS THE METRIC OF FLAT SPACETIME (EXPRESSED IN SPH COORDS OF COURSE)

- NOTE THAT l^a IS NULL, NOT ONLY W.R.T. FULL METRIC $g_{\mu\nu}$, BUT ALSO W.R.T. $\eta_{\mu\nu}$

$$\eta_{\mu\nu} l^\mu l^\nu = 0$$

- (**) GENERALIZES TO STATIONARY, AXISYMMETRIC SPACETIMES (KERR BH'S); l^μ NIGOMING NULL VEC W.R.T. BOTH $g_{\mu\nu}$ AND $\eta_{\mu\nu}$; $H = H(r, \theta)$

- BASIS FOR KERR-SCHILD COORDS; CURRENTLY SEEING HEAVY USE IN BBH GC EFFORT (PRL 80, 2512, 1998 gr-qc/9711078)

MASS ASPECT IN GENERAL SS SPACETIME

- CONSIDER OUR USUAL GENERAL FORM

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta\, dt\, dr + a^2 dr^2 + \underbrace{r^2 b^2 d\Omega^2}_{\equiv R^2 d\Omega^2}$$

- PROPER SURFACE AREA

$$A(r,t) = 4\pi R^2(r,t) \rightarrow \text{SCALAR ("GEOMETRIC QUANTITY")}$$

$$(\nabla^a A)(\nabla_a A) \quad \text{ALSO SCALAR}$$

- CONSIDER STATIC SCHWARZSCHILD METRIC

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \quad (*)$$

- IN SS, ANYWHERE ~~THAT~~ $T_{\mu\nu} = 0$ S.T. IS DESCRIBED BY A "PIECE" OF THIS SPACETIME. REGIONS WHERE $T_{\mu\nu} \neq 0$ WILL HAVE "MASS ASPECT", $m(r,t)$, "INTERPOLATING" BETWEEN $m(r,t) = M_1$ = CONSTANT AND $m(r,t) = M_2$ = CONSTANT

- NOW, FOR $(*)$ WE HAVE

$$A(R,t) = 4\pi R^2$$

$$\nabla_\mu A = (0, 8\pi R, 0, 0)$$

$$(\nabla^\mu A)(\nabla_\mu A) = g^{\mu\nu}(\nabla_\mu A)(\nabla_\nu A) = 64\pi^2 R^2 g^{RR}$$

$$= 16\pi A\left(1 - \frac{2M}{R}\right)$$

$$\Rightarrow \boxed{M(R) = \tfrac{1}{2} R\left(1 - (16\pi A)^{-1}(\nabla^\mu A)(\nabla_\mu A)\right)} \qquad (1)$$

OR, USING $A = 4\pi R^2$, WE FIND

$$\boxed{M(R) = \tfrac{1}{2} R\left(1 - (\nabla^\mu R)(\nabla_\mu R)\right)} \qquad (2)$$

CHECK: $M(R) = \frac{1}{2} R\left(1 - g^{\mu\nu}\nabla_\mu R \nabla_\nu R\right)$

$$= \tfrac{1}{2} R\left(1 - g^{RR}\right) = \tfrac{1}{2} R\left(1 - \left(1 - \frac{2M}{R}\right)\right)$$

$$= M$$

- EXPRESSIONS (1) AND (2) ARE CLEARLY SCALAR (R, $(\nabla^\mu R)(\nabla_\mu R)$ BOTH SCALARS); IMMEDIATELY GENERALIZE TO TIME DEPENDENT CASE

$$\boxed{\begin{aligned} m(r,t) &= \tfrac{1}{2} R\left(1 - (\nabla^\mu R)(\nabla_\mu R)\right) \\ &= \tfrac{1}{2}(rb)\left(1 - \nabla^\mu(rb)\,\nabla_\mu(rb)\right) \end{aligned}} \qquad (3)$$

MASS ASPECT IN IEF COORDINATES

- RECALL

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} $$

SO, IN PARTICULAR

$$ {}^{(4)}g^{rr} = {}^{(3)}g^{rr} - \beta^r\beta^r/\alpha^2 $$

$$ = a^{-2} - \alpha^{-2}\beta^2 $$

- NOW, IN IEF COORDINATES

$$ b(r,t) = 1 \quad , \quad \dot{b}(r,t) = 0 $$

$$ \alpha = a(1-\beta) \qquad \frac{\beta}{1-\beta} = raK^\theta_\theta $$

$$ \Rightarrow g^{rr} = a^{-2}\left(1 - \left(\frac{\beta}{1-\beta}\right)^2\right) = \left(a^{-2} - r^2 K^\theta_\theta{}^2\right) $$

- SO, MASS ASPECT IS

$$ m(r,t) = \frac{1}{2}(rb)\left(1 - \nabla^a(rb)\nabla_a(rb)\right) $$

$$ = \frac{1}{2}r\left(1 - {}^{(4)}g^{rr}\right) $$

$$\Rightarrow \quad \boxed{m(r,t) = \frac{1}{2} r \left(1 - a^{-2} + r^2 {K^\theta_\theta}^2\right)} \qquad (17)$$

* NOTE THAT AT LOCATION OF APPARENT HORIZON, r_{AH}, WE HAVE $H_{AH} = r_{AH}\, a K^\theta_\theta = 1$

$$\Rightarrow \quad m(r_{AH}, t) = \frac{r_{AH}}{2}$$

WHICH IS CONSISTENT WITH OUR EXPECTATIONS / ARGUMENT GIVEN ABOVE (BUT NOTE, CAN ONLY UNEQUIVOCALLY IDENTIFY $m(r_{AH}, t) = M_{BH}$ FOR $T_{\mu\nu} = 0$, $r \geq r_{AH}$)

* INSTRUCTIVE TO RE-EXPRESS (17) IN FORM INVOLVING MATTER VARIABLES

* MOMENTUM CONSTRAINT (13)

$$K^r{}_r = (r K^\theta{}_\theta)' + 4\pi r j_r$$

* HAMILTONIAN CONSTRAINT (12)

$$2ra' - a = -a^3\left(1 + r^2 K^\theta_\theta (K^\theta_\theta + 2K^r{}_r) - 8\pi r^2 \rho\right)$$

NOW CONSIDER $\left(r^3 {K^\theta_\theta}^2\right)' = \left[r (r K^\theta_\theta)^2 \right]'$

$$= r^2 {K^\theta_\theta}^2 + 2r^2 K^\theta_\theta (r K^\theta_\theta)'$$

$$= r^2 {K^\theta_\theta}^2 + 2r^2 K^\theta_\theta (K^r{}_r - 4\pi r j_r)$$

$$\Rightarrow r^2 K^\theta_\theta (K^\theta_\theta + 2K^r{}_r) = \left(r^3 {K^\theta_\theta}^2\right)' + 8\pi r^3 K^\theta_\theta j_r$$

• ALSO HAVE

$$2ra' - a = -a^3(ra^{-2})'$$

SO HAMILTONIAN CONSTRAINT BECOMES

$$-a^3(ra^{-2})' = -a^3\left(1 + (r^3 K^\theta{}_\theta{}^2)' - 8\pi r^2 \rho + 8\pi r^3 K^\theta_\theta j_r\right)$$

• DEFINING

$$\boxed{\sigma \equiv 4\pi r^2(\rho - r K^\theta_\theta j_r)}$$

$$\rho = \frac{\Phi^2 + \Pi^2}{2a^2} \qquad j_r = -\frac{\Phi \Pi}{a}$$

WE ~~HAVE~~

$$(ra^{-2})' = \left[r + (r^3 K^\theta{}_\theta{}^2) - \int^r 2\sigma \, d\tilde{r}\right]'$$

INTEGRATING

$$a^{-2} = 1 + r^2 K^\theta{}_\theta{}^2 - \frac{2}{r}\left(\underbrace{K}_{\text{INT. CONST.}} + \int^r \sigma \, d\tilde{r}\right)$$

THEN, FROM (17), WE HAVE

$$m(r,t) = \tfrac{1}{2} r\left(1 - a^{-2} + r^2 K^\theta{}_\theta{}^2\right)$$

$$= K + \int^r \sigma \, d\tilde{r}$$

- LOWER BOUND ON INTEGRAL IS $r = r_{AH}$, $m(r_{AH}, t) = r_{AH}/2$ FIXES CONSTANT K

$$\boxed{\begin{aligned} m(r,t) &= \frac{r_{AH}}{2} + \int_{r_{AH}}^{r} \sigma \, d\tilde{r} \\ &= \frac{r_{AH}}{2} + 4\pi \int r^2 \left(\rho - r K^\theta{}_\theta j_r\right) d\tilde{r} \end{aligned}}$$

NOTE THAT WE ALSO HAVE

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \left(\rho - r K^\theta{}_\theta j_r\right)}$$

NOT $\dfrac{dm}{dr} = 4\pi r^2 \rho$ AS IN P.A. CASE !

PROBLEMS WITH IEF COORDINATES

- IEF INCOMPATIBLE WITH REGULARITY / LOCAL FLATNESS AT $r=0$ (SEE SECTION III OF MARSA : CHOPTUIK); HEURISTICALLY CAN THINK OF BH SPACETIME IN LIMIT AS $m \to 0$ AS BEING DISTINCT FROM FLAT SPACETIME (ALTHOUGH AT LARGE SCALES $L \gg m$, ST'S ARE PHYSICALLY IDENTICAL)

- SOLUTION - IF INTERESTED IN STUDYING BH FORMATION WITH SUBSEQUENT EXCISING => USE AH-PENETRATING COORDINATES WHICH ARE COMPATIBLE WITH A REGULAR ORIGIN EXAMPLES: MAXIMAL / ISOTROPIC

  MAXIMAL / AREAL ~> MARSA et al WORK IN PROGRESS

In a vacuum region of a spherically symmetric spacetime (SSST), adopt the usual Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2. \tag{1}$$

Then, the proper surface area at radius r is $\mathcal{A}(r) = 4\pi r^2$, and is a geometric quantity as are

$$\widetilde{d\mathcal{A}} = 8\pi r \widetilde{dr} \tag{2}$$

and

$$\widetilde{d\mathcal{A}} \cdot \widetilde{d\mathcal{A}} = 64\pi^2 r^2 \widetilde{dr} \cdot \widetilde{dr} = 16\pi \mathcal{A} g^{rr}. \tag{3}$$

Thus, we have

$$M(r) = \frac{1}{2} r \left(1 - (16\pi\mathcal{A})^{-1} \mathcal{A}^{,\mu}\, \mathcal{A}_{,\mu}\right). \tag{4}$$

Now, first consider the general 3+1 form of the the 4–metric for SSST—(the r, t coordinates here are, of course, different from those of (1)):

$$ds^2 = \left(-\alpha^2 + a^2\,\beta^2\,\right) dt^2 + 2a^2\,\beta dt\, dr + a^2\, dr^2 + r^2\, b^2\, d\Omega^2. \tag{5}$$

we wish to introduce a "shifted" areal coordinate, s, such that $s = r + f(t)$ and the metric takes the form:

$$ds^2 = \left(-\alpha^2 + a^2\,\beta^2\,\right) dt^2 + 2a^2\,\beta dt\, dr + a^2\, dr^2 + s^2\, d\Omega^2. \tag{6}$$

So

$$b = 1 + \frac{f}{r} \tag{7}$$

and

$$\dot{(rb)} = \dot{f} \qquad\qquad (rb)' = 1 \tag{8}$$

Using these last relations, the evolution equation for the 3–metric function, b,

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}\,(br)'\,,$$

trivially rewritten as

$$\dot{(rb)} = -\alpha\,(rb)\,K^\theta{}_\theta + \beta\,(br)'\,, \tag{9}$$

is solved for β, giving the spatial coordinate condtion:

$$\beta = \dot{f} + s\alpha K^\theta{}_\theta. \tag{10}$$

The time–slicing relation comes from considering the tangent vectors:

$$\frac{\vec{\partial}}{\partial t} = (1, 0, 0, 0) \qquad\qquad \frac{\vec{\partial}}{\partial r} = (0, 1, 0, 0)$$

and demanding that the *ingoing* combination, $\overrightarrow{\partial/\partial t} - \overrightarrow{\partial/\partial r}$, be null, which in SSST also implies that the vector is tangent to an ingoing null geodesic. Thus,

$$g_{tt} - 2g_{tr} + g_{rr} = 0, \tag{11}$$

and using (5), this implies,

$$\alpha = a(1 - \beta), \tag{12}$$

where the particular solution of the quadratic has been chosen to give a positive lapse for $|\beta| < 1$. Finally, we can rewrite the coordinate conditions by combining (10) and (12), giving

$$\beta = \frac{\dot{f} + Q}{1 + Q} \tag{13}$$

$$\alpha = a\left(\frac{1 - \dot{f}}{1 + Q}\right) \tag{14}$$

where

$$Q = saK^{\theta}{}_{\theta} \tag{15}$$

We now consider the structure of (4) in this coordinate system. We have

$$\mathcal{A} = 4\pi s^2 \qquad \mathcal{A}_{,t} = 8\pi s\dot{f} \qquad \mathcal{A}_{,r} = 8\pi s. \tag{16}$$

So,

$$\begin{aligned}
(16\pi\mathcal{A})^{-1}\mathcal{A}^{,\mu}\mathcal{A}_{,\mu} &= (16\pi\mathcal{A})^{-1}\left(g^{tt}\mathcal{A}_{,t}{}^2 + 2g^{tr}\mathcal{A}_{,t}\mathcal{A}_{,r} + g^{rr}\mathcal{A}_{,r}{}^2\right) \\
&= -\alpha^{-2}\dot{f}^2 + 2\beta\alpha^{-2}\dot{f} + a^{-2} - \beta^2\alpha^{-2} \\
&= a^{-2} - \alpha^{-2}\left(\dot{f} - \beta\right)^2 \\
&= \frac{1 - 2\beta - \dot{f}\left(\dot{f} - 2\beta\right)}{a^2\left(1 - \beta\right)^2}
\end{aligned} \tag{17}$$

However, using (13) and (14), we find

$$(1 - \beta)^{-2}\left(1 - 2\beta - \dot{f}\left(\dot{f} - 2\beta\right)\right) = 1 - Q^2 \tag{18}$$

So, the final expression for the mass in these coordinates is

$$M = \frac{1}{2}s\left(1 - \frac{1 - Q^2}{a^2}\right). \tag{19}$$

For the static Schwarzschild case, where $Q = raK^{\theta}{}_{\theta}$, this becomes

$$M = \frac{1}{2}r\left(\frac{a^2 - 1}{a^2} + r^2 K^{\theta}{}_{\theta}{}^2\right). \tag{20}$$

But, in these coordinates, we have the following expressions for the (static) 3+1 variables:

$$\alpha = \sqrt{\frac{r}{(r + 2M)}}, \tag{21}$$

$$\beta = \frac{2M}{(r + 2M)}, \tag{22}$$

$$a = \sqrt{\frac{(r + 2M)}{r}}, \tag{23}$$

$$K^{r}{}_{r} = \frac{-2M\,(r + M)}{(r\,(r + 2M))^{\frac{3}{2}}}, \tag{24}$$

$$K^{\theta}{}_{\theta} = \frac{2M\,(r + 2M)}{(r\,(r + 2M))^{\frac{3}{2}}}. \tag{25}$$

So,

$$\frac{a^2-1}{a^2}=\frac{2M}{(r+2M)} \qquad r^2\, {K^\theta}_\theta{}^2=\frac{(2M)^2}{r\left((r+2M)\right)}, \tag{26}$$

and we see that the left and right hand sides of (20) are indeed equal.

Let us now consider the job of tracking an apparent horizon in this coordinate system. The equation for a trapped surface, defined by an outward–pointing, space–like unit normal, s^μ, is (see, for example, section 2.5 of my PhD thesis):

$$s^i{}_{|i}-K+s^i s^j K_{ij}=0, \tag{27}$$

which, for the general spherically–symmetric metric (5), and

$$K^i{}_j=\mathrm{diag}(K^r{}_r, K^\theta{}_\theta, K^\theta{}_\theta), \tag{28}$$

becomes:

$$(rb)'-arbK^\theta{}_\theta=0. \tag{29}$$

It is convenient at this point to write down the various 3+1 Einstein equations that are available to us. The Hamiltonian and momentum constraints are

$$a'-\frac{1}{2s}\left(a-a^3\right)-\frac{1}{2}s\left(4\pi a\left(\Phi^2+\Pi^2\right)-a^3K^\theta{}_\theta\left(K^\theta{}_\theta+2K^r{}_r\right)\right)=0 \tag{30}$$

and

$$K^\theta{}_\theta{}'+\frac{K^\theta{}_\theta-K^r{}_r}{s}-\frac{4\pi\Phi\Pi}{a}=0. \tag{31}$$

We also have evolution equations for a, $K^r{}_r$, and $K^\theta{}_\theta$. The following forms of these incorporate the slicing condition, (12), but not relations (13) and (14):

$$\dot{a}=-a^2\left(1-\beta\right)K^r{}_r+(a\beta)' \tag{32}$$

$$\dot{K}^\theta{}_\theta=\beta K^\theta{}_\theta{}'+\frac{1-\beta}{s^2}\left(a-\frac{1}{a}\right)+\frac{\beta'}{as}+a\left(1-\beta\right)K^\theta{}_\theta\left(2K^\theta{}_\theta+K^r{}_r\right) \tag{33}$$

$$\dot{K}^r{}_r=\beta K^r{}_r{}'+\frac{\beta-1}{a}\left(\frac{a''}{a}-\left(\frac{a'}{a}\right)^2-\frac{2a'}{sa}+8\pi\Phi^2\right)+\frac{\beta'a'}{a^2}+\frac{\beta''}{a}+a\left(1-\beta\right)K^r{}_r\left(2K^\theta{}_\theta+K^r{}_r\right). \tag{34}$$

Also, we have the evolution equations for the scalar field variables Φ and Π:

$$\dot{\Phi}=\left(\beta\Phi+\left(1-\beta\right)\Pi\right)' \tag{35}$$

$$\dot{\Pi}=\frac{1}{s^2}\left(s^2\left(\beta\Pi+\left(1-\beta\right)\Phi\right)\right)'-2\frac{\dot{s}}{s}\Pi \tag{36}$$

where

$$\Phi\equiv\phi' \qquad \Pi\equiv\frac{a}{\alpha}\left(\dot{\phi}-\beta\phi'\right), \tag{37}$$

as usual. Now, let us define the function H as follows:

$$H(r,t)=(rb)'-arbK^\theta{}_\theta. \tag{38}$$

Then, in the current coordinate system,

$$H=1-asK^\theta{}_\theta=1-Q. \tag{39}$$

Clearly then, $Q(r^\star,t)=1$ means that there is an apparent horizon at $r^\star$ at time t, and if we wish to tie the edge of a numerical grid (which, it will be assumed, will have a fixed r label) to the apparent horizon, we must impose

$$\dot{Q}\big|_{r=r^\star}=0 \tag{40}$$

for all t. If we differentiate $asK^\theta{}_\theta = 0$ with respect to time, use the evolution equations to eliminate the time derivatives and the constraint equations to eliminate a' and $K^\theta{}_\theta{}'$, then solve for $\dot{f}$ and use $Q(r^\star, t) = 1$, we find (somewhat amazingly)

$$\dot{f} = 4\pi s^2 \frac{\Sigma^2}{a^2}\bigg|_{r=r^\star} \tag{41}$$

where

$$\Sigma \equiv \Phi + \Pi \qquad \Theta \equiv \Phi - \Pi \tag{42}$$

will be useful definitions here and below. We can also derive equations of motion for the geometric variables at $r^\star$, again using the evolution and constraint equations as well as the known values for $Q(r^\star, t)$ and $\dot{f}(t)$. We find (after some problems with the 84 version of Reduce running under CMS)

$$\dot{a}\Big|_{r=r^\star} = -\frac{3}{8}\frac{a^3}{s} - a^2 K^r{}_r + \pi s a\left(\Phi^2 + \Pi^2\right) + 2\pi^2 \frac{s^3}{a}\Sigma^2\Theta^2\bigg|_{r=r^\star} \tag{43}$$

$$\begin{aligned} \dot{K}^\theta{}_\theta\Big|_{r=r^\star} &= \frac{3}{8}\frac{a}{s^2} + \frac{K^r{}_r}{s} - \pi\frac{\Phi^2 + \Pi^2}{a} - 2\pi\frac{\left(2 + \pi s^2\Theta^2\right)\Sigma^2}{a^3}\bigg|_{r=r^\star} \\ &= -\left(\frac{\dot{a}}{sa^2} + \frac{\dot{f}}{s^2 a}\right)\bigg|_{r=r^\star} \end{aligned} \tag{44}$$

The last result is trivially derivable from $\dot{Q} = 0$, $Q(r^\star, t) = 1$.

Let us do some manipulations with the constraint equations. The momentum constraint, (31), is easily solved for $K^r{}_r$,

$$K^r{}_r = \left(sK^\theta{}_\theta\right)' - 4\pi s\frac{\Phi\Pi}{a}, \tag{45}$$

and the Hamiltonian constraint may be written as

$$2sa' - a = -a^3\left(1 + s^2 K^\theta{}_\theta\left(K^\theta{}_\theta + 2K^r{}_r\right) - 4\pi s^2\frac{\Phi^2 + \Pi^2}{a^2}\right). \tag{46}$$

Now, using (45), it is easy to show that

$$s^2 K^\theta{}_\theta\left(K^\theta{}_\theta + 2K^r{}_r\right) = \left(s^3 {K^\theta{}_\theta}^2\right)' - 8\pi s^3 K^\theta{}_\theta\frac{\Phi\Pi}{a}. \tag{47}$$

Thus, (46) becomes

$$-a^3\left(sa^{-2}\right)' = -a^3\left(s + s^3 {K^\theta{}_\theta}^2 - \int^s 2\sigma\, ds\right)' \tag{48}$$

where

$$\begin{aligned} \sigma &\equiv 2\pi s^2\frac{\Phi^2 + \Pi^2}{a^2} + 4\pi s^3 K^\theta{}_\theta\frac{\Phi\Pi}{a} \\ &= 4\pi s^2\left(\rho - s\,K^\theta{}_\theta\, j_r\right). \end{aligned} \tag{49}$$

Integrating the above, we find

$$a^{-2} = 1 + s^2 {K^\theta{}_\theta}^2 - \frac{2}{s}\left(\kappa + \int^s \sigma\, ds\right) \tag{50}$$

where κ is a constant of integration. Now, multiplying by a^2 and rearranging we find, recalling (15),

$$\frac{1 - Q^2}{a^2} = 1 - \frac{2}{s}\left(\kappa + \int^s \sigma\, ds\right). \tag{51}$$

The value of the integration constant may be deduced by recalling that we are requiring $Q(r^\star, t) = 1$ and are considering outwards integration. Thus

$$\kappa = \frac{s^\star}{2} = M^\star \tag{52}$$

Also, we have another expression for the mass

$$M = M^\star + \int^s \sigma\, ds = \frac{s^\star}{2} + 4\pi\int s^2\left(\rho - s\,K^\theta{}_\theta\, j_r\right) ds. \tag{53}$$