SOLVING THE 3+1 EQNS: OVERVIEW

- 3+1 VARIABLES

	COORD-FREE	IN $\{t, x^i\}$ C.S.	#
(1) "KINEMATICAL":	α	α	1
	β^a	β^i	3
			4
(2) "DYNAMICAL":	γ_{ab}	γ_{ij}	6
	$K^a{}_b$	$K^i{}_j$	6
			12

- SCHEMATIC FOR SOL^N of 3+1 EQNS ("CONSTRUCTION of SPACETIME")

(1) DECIDE ON COORDINATE SYSTEM; I.E. HOW α, β^i ARE TO BE CHOSEN (ALSO SEE STEPS 2, 3) - WILL ASSUME THROUGHOUT THAT OUR SINGLE CHOICE WILL COVER ENTIRE S.T. (QUITE POSSIBLY UNREALISTIC FOR SOME [MANY?] PHYSICALLY INTERESTING SCENARIOS)

(2) CHOOSE $t=0$ (INITIAL) SLICE $\Sigma(0)$ - INCLUDING SPECIFICATION of TOPOLOGY of $\Sigma(0)$ (M WILL, BY ASSUMPTION, HAVE TOPOLOGY $\mathrm{TOP}(\Sigma(0)) \times \mathbb{R}$), INITIAL SPATIAL COORDINATIZATION

(3) SPECIFY INITIAL DATA $\{\gamma_{ij}(x^k,0),\ K^i{}_j(x^k,0)\}$ $\{\rho(x^k,0),\ j^i(x^k,0)\}$ SUBJECT TO CONSTRAINTS

$$R - K^2 + K^i{}_j K_i{}^j = 16\pi\rho$$

$$D_i K^{ij} - D^j K = 8\pi j^i$$

AND, IF NECESSARY, COMPATIBLE WITH COORDINATE CHOICES (α, β^i PRESCRIPTIONS)

(4) EVOLVE DATA TO FUTURE (AND/OR PAST) USING

$$\mathring{\gamma}_{ij} = \ldots$$

$$\mathring{K}^i{}_j = \ldots \qquad + \text{ E.O.M FOR MATTER FIELDS}$$

AND POSSIBLY THE CONSTRAINT EQUATIONS TO GENERATE

$$\{ \gamma_{ij}(x^k, t), \ K^i{}_j(x^k, t) \}$$

AND, IMPLICITLY,

$$\{ \rho(x^k, t), \ j^i(x^k, t), \ S^i{}_j(x^k, t) \}$$

STAGE 3: THE INITIAL-VALUE PROBLEM FOR GR (IVP)

STAGE 4: THE EVOLUTION PROBLEM FOR GR. (EP)

STAGES 1 ; 2 BEAR ON BOTH THE IVP ; EP ; ALSO, SOLN of 3+1 EQNS = CAUCHY PROBLEM FOR GR

WILL DISCUSS SOME GENERAL ISSUES, THEN EXAMINE THE IVP AND EP, IN TURN, AND IN SOME DETAIL

- DYNAMICAL / CONSTRUCTIVE / CAUCHY APPROACH: "UPDATE PARADIGM" USEFUL (VACUUM CASE ILL. HERE)

LET $G(x^i,t)$ BE A COMPLETE SPECIFICATION OF THE 4-GEOMETRY AT TIME t:

$$G(x^i,t) \equiv \{\gamma_{ij},\ K^i{}_j\ ;\ \alpha,\ \beta^i\}(x^k,t)$$
$$\equiv G_I(x^i,t)\ ,\quad I = 1, 2, \ldots$$

"UPDATE PARADIGM"

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G(x^k, 0) := INITIAL DATA
t := 0
DO UNTIL t >= t_DESIRED
    FOR EACH CONSTITUENT FIELD G_I DO:
        G_I(x^k, t+dt) := UPDATE_G_I [G(x^k, t)]
    END FOR
    t := t + dt
END DO
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- NOTE THAT IN THIS VIEW KINEMATICAL VBL^S AS WELL AS DYNAMICAL VBL^S ARE UPDATED

- IN GENERAL MANY ALTERNATIVES FOR CONSTRUCTING SUFFICIENT SET OF UPDATE OPERATORS, LARGELY DUE TO EXISTENCE OF CONSTRAINT EQNS AS WELL AS EVOLUTION EQUATIONS ("OVER-DETERMINED" SYSTEM — MORE EQN^S THAN UNKNOWNS)

CHOICE of TOPOLOGY of $\Sigma(0)$

- AGAIN NOTE THAT $G_{ab} = 8\pi T_{ab}$ HAS NOTHING TO SAY ABOUT TOPOLOGY of Σ_t — COULD BE $\mathbb{R}^3$, S^3, $S^2 \times \mathbb{R}$ etc.

- SOL^N of CONSTRAINT EQN'S PARTICULARLY SENSITIVE TO THIS ISSUE SINCE CE'S TYPICALLY SOLVED AS BOUNDARY VALUE PROBLEM

- UNLESS ~~OTHERWISE~~ SPECIFIED, WE WILL RESTRICT ATTN TO Σ_t WITH $\mathbb{R}^3$ TOPOLOGY, CHIEF RATIONALE: MOST LOGICALLY COMPATIBLE WITH ASYMPTOTIC FLATNESS

ASYMPTOTIC FLATNESS (A.F.)

- INTUITIVE NOTION: METRIC TENDS TO FLAT-METRIC AT LARGE DISTANCES

- AF SPACETIMES LIKELY TO BE PARTICULARLY RELEVANT PHYSICALLY FOR STUDY of ISOLATED SELF-GRAVITATING SYSTEMS (BUT KEEP IN MIND ~~THAT~~ AF ALSO LIKELY ONLY TO BE APPROXIMATE IN PRACTICE, DUE TO LARGE-SCALE [COSMOLOGICAL] CURVATURE)

- SLIGHTLY MORE PRECISE DEFN (SEE WALD CH 11 FOR MUCH MORE RIGOR / DETAILS / POINTERS TO LITERATURE)

ASSUME WE CAN FIND CARTESIAN-LIKE SPATIAL COORDINATES $\{x, y, z\}$; DEFINE $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

DEMAND THAT AS $r \to \infty$, HAVE

$$\gamma_{ij} = f_{ij} + h_{ij} = f_{ik}\left(\delta^k{}_j + h^k{}_j\right)$$

$$f_{ij} = \mathrm{diag}(1, 1, 1)$$

SUCH THAT THE FOLLOWING <u>FALL-OFF</u> CONDITIONS ARE SATISFIED (AGAIN, AS $r \to \infty$)

$$h^i{}_j = O(r^{-1})$$

$$\partial_x h^i{}_j = O(r^{-2})$$

$$\partial_y h^i{}_j = O(r^{-2})$$

$$\partial_z h^i{}_j = O(r^{-2})$$

OR LIE DERIVATIVE OF $h^i{}_j$ ALONG TRANS-LATIONAL KILLING VECTORS $k^i{}_{(j)}$ $j = 1, 2, 3$ $= O(r^{-2})$

$$\mathcal{L}_{k(j)} h^i{}_j = O(r^{-2})$$

$$K^i{}_j = O(r^{-2})$$

ENERGY (MASS) AND MOMENTUM (ADM)

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \oint D^j\left(h^i{}_j - \delta^i{}_j h\right) d^2S_i$$

$$P_j k^j{}_{(i)} = \lim_{r\to\infty} \frac{1}{8\pi} \oint \left(K^m{}_j - \delta^m{}_j K\right) k^j{}_{(i)}\, d^2S_m$$

- CAN LOOSELY THINK OF E AND P_j AS COEFFICIENTS IN (ASYMPTOTIC, $r \to \infty$) MULTIPOLE EXPANSION of γ_{ij}, $K^i{}_j$; SURFACE INTEGRALS AT SPATIAL INFINITY "PICK-OFF" COEFFICIENTS

"COUNTING ARGUMENTS" — # of TRUE DYNAMICAL DEGREES OF FREEDOM IN GEOMETRODYNAMICS

3+1 VBLS

KINEMATICAL: α, β^i (4)

DYNAMICAL: $\gamma_{ij}, K^i{}_j$ (12)

E.O.M.

CONSTRAINT: $R + K^2 - K^i{}_j K_i{}^j = 16\pi\rho$ (4)

$D_j K^{ij} - D^i K = 8\pi j^i$

EVOLUTION: $\mathring{\gamma}_{ij} = \ldots, \mathring{K}^i{}_j = \ldots$ (12)

CONSIDER OUR "UPDATE PARADIGM" FOR VACUUM CASE (NO REAL L.O.G.)

(1) MUST GIVE PRESCRIPTION FOR FIXING α, β^i; WILL TREAT THIS ISSUE IN SOME DETAIL — HERE WE CONSIDER SOME EXAMPLES

(a) $\alpha(x^k, t) = 1 \qquad \beta^i(x, t) = 0$

→ "SYNCHRONOUS COORDINATES": COORDINATE STATIONARY OBSERVERS (CSO's) FLOW NORMAL TO HYPERSURFACES, + LABELS PROPER TIME of CSO's

⇒ CSO's ARE GEODESIC ($t^a = n^a$;

$a_b = n^a \nabla_a n_b$

$= D_b \ln \alpha = 0$)

NOTE THAT, GIVEN A SPACETIME, WILL HAVE INFINITE # OF $\{\alpha = 1, \beta^i = 0\}$ FOLIATIONS SINCE THE PRESCRIPTION DOES NOT FIX AN INITIAL SLICE

(b) $\beta^i = 0$, $\alpha =$ "SOMETHING ELSE" $\to$ "NORMAL (SPATIAL) COORDINATES". HISTORICALLY, OFTEN CHOSEN DUE TO SIMPLIFICATION OF EVOLUTION EQUATIONS ($\mathcal{L}_\beta$ TERMS DROP OUT)

(c) α OFTEN CHOSEN VIA CONDITION ON EXTRINSIC CURVATURE, ESPECIALLY ON $\mathrm{Tr}\,K = K^i{}_i$

- RECALL EV. EQN FOR $K^a{}_b$

$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha$$
$$+ \alpha\left(R^a{}_b + K K^a{}_b + 8\pi\left(\tfrac{1}{2}\delta^a{}_b(S-\rho) - S^a{}_b\right)\right)$$

TAKE TRACE

$$(*) \qquad \mathcal{L}_t K = \mathcal{L}_\beta K - D^a D_a \alpha + \alpha\left(R + K^2 + 8\pi\left(\tfrac{1}{2}S - \tfrac{3}{2}\rho\right)\right)$$

- NOW, SUPPOSE WE DEMAND THAT

$$\mathrm{Tr}\,K = K^i{}_i = K = K(x^k, t) = 0$$

WHICH IS CALLED MAXIMAL SLICING (SINCE $K = 0 \Rightarrow$ VOLUME of Σ IS MAXIMIZED W.R.T. INF. DEFORMATIONS OF Σ)

OPERATIONALLY, THIS CAN BE IMPLEMENTED BY DEMANDING THAT

$$K(x^k, 0) = 0$$

$$\dot{K}(x^k, t) = \partial_t K(x^k, t) = \mathcal{L}_t K(x^k, t) = 0$$

FROM (*), THIS LAST CONDITION BECOMES THE FOLLOWING ELLIPTIC P.D.E FOR $\alpha(x^k, t)$

$$\boxed{D^a D_a \alpha = \alpha\left(R + 8\pi\left(\tfrac{1}{2}S - \tfrac{3}{2}\rho\right)\right)}$$

WHICH CAN BE SOLVED AS A B.V.P. SUBJECT TO $\alpha \to 1$ AS $r \to \infty$ (ASYMPTOTIC FLATNESS + COORD TIME AT ∞ IS NORMALIZED TO PROPER TIME)

OBSERVE THAT $K=0$ NOT ONLY FIXES OUR FOLIATION (CHOICE of SLICING) — ONCE SUPPLEMENTED WITH SOME INITIAL SLICE COMPATIBLE WITH $K=0$ — IT ALLOWS US TO ELIMINATE ONE DYNAMICAL VBL FROM THE E.O.M.; FOR EXAMPLE, ASSUMING $\{t, x^i\} = \{t, x, y, z\}$

$$K = K^x{}_x + K^y{}_y + K^z{}_z$$

$$K = 0 \Rightarrow K^z{}_z = -(K^x{}_x + K^y{}_y)$$

CLEARLY CAN USE THIS TO ELIMINATE $K^z{}_z$ FROM E.O.M.

(2) ROLE of CONSTRAINTS VIZ A VIZ "UPDATE PARADIGM"

CONSTRAINTS: $\mathcal{H}^a \equiv (G^{ab} - 8\pi T^{ab}) n_a = 0$

$$\mathcal{H}^a(x^k, 0) = 0 \quad + \quad \perp (G^{ab} - 8\pi T^{ab})(x^k, t) = 0$$

$$\Rightarrow \mathcal{H}^a(x^k, t) = 0$$

I.E. IF CONSTRAINTS HOLD ~~AT~~ $t=0$, AND EV. EQNS ARE SATISFIED EVERYWHERE THEN THE CONSTRAINTS ARE SATISFIED EVERYWHERE

- BUT, OPERATIONALLY / CONSTRUCTIVELY COULD RESOLVE $\mathcal{H}^a = 0$, AT EACH TIME STEP, FOR UPTO 4 OF $\{\gamma_{ij}, K^i{}_j\}$, THUS ELIMINATING NEED FOR USE of CORRESPONDING EV. EQNS $\dot{\gamma}_{ij} = \ldots$, $\dot{K}^i{}_j = \ldots$

- THUS, CAN IN PRINCIPLE USE

(1) COORDINATE FREEDOM (COORDINATE CONDITIONS)

(2) CONSTRAINT EQUATIONS

TO ELIMINATE (NEED TO USE) UPTO 8 OF THE EVOLUTION EQUATIONS

- IN THE CASE WHERE THE MAX # (8) of EV. EQNS

ARE ELIMINATED (WHICH MUST BE PHYSICALLY EQUIVALENT TO ANY AND ALL OTHER SCHEMES) WE ARE LEFT WITH $12 - 8 = 4$ FIRST-ORDER EV. EQN'S $\equiv$ 2 SECOND-ORDER E.O.M.

$\boxed{\Rightarrow \text{ GR HAS 2 DYNAMICAL DEGREES of FREEDOM}}$

<u>NOTE</u>: DYNAMICAL DEGREE of FREEDOM $\equiv$ <u>UNCONSTRAINED</u> (q, π) PAIR; CONSTRAINTS WILL GENERALLY APPLY TO <u>EITHER</u> q OR π

<u>NOTE</u>: TO REDUCE # OF EV EQN'S REQUIRED TO MINIMUM - 4 - HAD TO USE <u>FULL</u> COORD FREEDOM TO <u>ELIMINATE</u> 4 OF γ_{ij}, $K^i{}_j$; IN GENERAL, MAY <u>NOT</u> WANT TO DO THIS (E.G. $\alpha = 0, \beta = 1$); IN ADDITION MAY <u>NOT</u> WANT TO SOLVE CONSTRAINTS AT EACH TIME STEP

$\Rightarrow$ CLEARLY, MANY DISTINCT UPDATE SCHEMES WILL BE POSSIBLE IN GENERAL — ONE of KEY FEATURES of SOL'N OF 3+1 EQN'S

<u>FINAL NOTE</u>: COUNTING MODIFIED WHEN SYMMETRIES IMPOSED, E.G. SPHERICAL SYMMETRY, MOST GENERAL FORM

$$\alpha = \alpha(r,t) \qquad \beta^i = (\beta^r(r,t), 0, 0)$$

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t), r^2 b^2(r,t), r^2 b^2 \sin^2\theta\right)$$ * $\Rightarrow$ NO DYNAMICAL DEGREES of FREEDOM!

$$K^i{}_j = \mathrm{diag}\left(K^r{}_r(r,t), K^\theta{}_\theta(r,t)\right)$$

4 EV EQNS $(\dot a, \dot b, \dot K^r{}_r, \dot K^\theta{}_\theta)$ BUT CAN ELIMINATE 2 OF $\{a, b, K^r{}_r, K^\theta{}_\theta\}$, USING COORD FREE, SOLVE FOR OTHER 2 FROM CONSTRAINTS *