PHY 387N HAMILTONIAN FORM of CLASSICAL F.T.

(WALD E.2)

- HAVE CAST EINSTEIN EQN'S INTO FIRST-ORDER-IN TIME FORM (3+1 FORM)

$$\partial_t \gamma_{ij} = \mathcal{L}_t \gamma_{ij} - \cdots$$

$$\partial_t K^i{}_j = \mathcal{L}_t K^i{}_j - \cdots$$

- WITHIN 3+1 CONTEXT, WILL GENERALLY BE CONVENIENT TO CAST E.O.M. FOR ANY OTHER FIELDS (MATTER FIELDS) IN SAME FORM $\Rightarrow$ HAMILTONIAN FORMULATION

- RECALL LAGRANGIAN APPROACH

ψ: FIELD CONFIGURATION (INDICES SUPRESSED)

$$\mathcal{L} = \mathcal{L}(\psi, \nabla_a\psi, \nabla_a\nabla_b\psi, \ldots) \rightarrow \text{LAGRANGIAN DENSITY}$$

$$= \sqrt{-g}\, L(\psi, \nabla_a, \nabla_a\nabla_b\psi, \ldots) \rightarrow \text{LAGRANGIAN SCALAR}$$

$S[\psi]$: ACTION FUNCTIONAL

$$= \int \mathcal{L}\, d^4x = \int L \sqrt{-g}\, d^4x = \int L\, dV$$

$$0 = \delta S = \left.\frac{dS}{d\lambda}\right|_{\lambda=0} \Rightarrow \text{FIELD EQUATIONS (E.O.M)}$$

E.G. IF $\mathcal{L} = \mathcal{L}(\psi, \nabla_a \psi)$ WILL GET 2ND-ORDER "EULER-LAGRANGE" EQUATIONS

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = \frac{\partial \mathcal{L}}{\partial \psi}$$

HAMILTONIAN APPROACH

$\psi \to q$: CONFIGURATION VARIABLE

$\mathcal{L}_t \psi \equiv \partial_t \psi \equiv \dot{\psi} \rightsquigarrow \pi$: (CONJUGATE) MOMENTUM VBL

REWRITE EQUATIONS OF MOTION IN FORM

$$\mathcal{L}_t\, q \equiv \dot{q} \equiv \partial_t q = \frac{\delta H}{\delta \pi} \qquad (E.2.2)$$

$$\mathcal{L}_t\, \pi \equiv \dot{\pi} \equiv \partial_t \pi = -\frac{\delta H}{\delta q} \qquad (E.2.3)$$

WHERE $$H[q,\pi] = \int_{\Sigma(t)} \mathcal{H}(q,\pi)\, d^3x$$

AND $\mathcal{H}(q,\pi)$ IS THE HAMILTONIAN DENSITY CONSTRUCTED SO THAT (E.2.2), (E.2.3) ARE EQUIVALENT TO THE LAGRANGIAN FIELD EQUATIONS

* GIVEN A LAGRANGIAN FORM., HAM. FORM. CAN BE OBTAINED VIA FOLLOWING PROCESS - COMPLETELY ANALOGOUS TO LAG → HAM IN CLASSICAL MECHANICS

(1) $q \equiv \psi$ (AGAIN, THIS NOTATION SUPPRESSES ALL INDICES)

(2) ASSUMING $\mathcal{L}$ DOES NOT DEPEND ON TIME DERIVATIVES HIGHER THAN 1ST ORDER (E.G. $\mathcal{L} = \mathcal{L}(\psi, \nabla_a \psi)$), DEFINE CONJUGATE MOMENTUM

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad \text{(E.2.4)}$$

NOTE: π WILL, IN GENERAL, BE A DENSITY (I.E. WILL BE PROPORTIONAL TO $\sqrt{-g}$)

(3) SOLVE (E.2.4) FOR $\dot{q} = \dot{q}(q, \pi)$; ASSUMING EXPLICIT SOL^{N} IS POSSIBLE, DEFINE HAMILTONIAN DENSITY:

$$\mathcal{H}(q, \pi) = \pi \dot{q} - \mathcal{L} \qquad \text{(E.2.5)}$$

IMPLIED SUMMATION OVER SUPPRESSED FIELD INDICES

(4) FIRST ORDER E.O.M. COMPUTED VIA VARIATION of HAMILTONIAN $H(q, \pi) = \int_\Sigma \mathcal{H}(q, \pi)\, d^3x$

$$\dot{q} = \frac{\delta H}{\delta \pi}$$

$$\dot{\pi} = -\frac{\delta H}{\delta q}$$

WILL THEN BE EQUIVALENT TO LAG. E.O.M $\left.\frac{\delta S}{\delta \mathcal{U}}\right|_{\mathcal{U}} = \left.\frac{dS}{d\lambda}\right|_{\lambda=0} = 0$

PROOF: TAKE

$$J = \int_{t_1}^{t_2} H \, dt = \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \mathcal{H} \, d^3x$$

$$= - \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \mathcal{L} \, d^3x + \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \pi \dot{q} \, d^3x$$

$$= - S + \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \pi \dot{q} \, d^3x$$

NOW CONSIDER A 1-PAR VAR $\mathcal{U}_\lambda$ SATISFYING $\delta \mathcal{U}|_{t=t_1, t_2} = 0$

$$\Rightarrow \delta J \equiv \left.\frac{dJ}{d\lambda}\right|_{\lambda=0} = \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \left(\frac{\delta H}{\delta q} \delta q + \frac{\delta H}{\delta \pi} \delta \pi \right) d^3x \qquad (*)$$

$$= -\delta S + \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \left(\pi \, \delta \dot{q} + \dot{q} \, \delta \pi \right) d^3x$$

$$= -\delta S + \int_{t_1}^{t_2} dt \int_{\Sigma(t)} \left(-\dot{\pi} \, \delta q + \dot{q} \, \delta \pi \right) d^3x \qquad (**)$$

THUS, COMPARING $(*)$ AND $(**)$, WE SEE THAT THE LAG E.O.M, $\delta S = 0$, ARE SATISFIED IFF

$$\dot{q} = \frac{\delta H}{\delta \pi} \qquad \dot{\pi} = - \frac{\delta H}{\delta q}$$

Q.E.D.

EXAMPLE: HAMILTONIAN (3+1) E.O.M. FOR MASSLESS, NON SELF-INTERACTING, SCALAR FIELD IN CURVED S.T.

$$\mathcal{L}_{MKG} = -\frac{1}{2}\sqrt{-g}\, g^{ab}\phi_{,a}\phi_{,b}$$

$$= -\frac{1}{2}\sqrt{-g}\left(g^{00}\phi_{,0}^{2} + 2g^{0i}\phi_{,0}\phi_{,i} + g^{ij}\phi_{,i}\phi_{,j}\right)$$

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} \qquad \sqrt{-g} = \alpha\sqrt{\gamma}$$

$$\therefore \mathcal{L}_{MKG} = -\frac{1}{2}\alpha\gamma^{\frac{1}{2}}\left(-\alpha^{-2}\phi_{,0}^{2} + 2\alpha^{-2}\beta^i\phi_{,i}\phi_{,0} + \left(\gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2}\right)\phi_{,i}\phi_{,j}\right)$$

- COMPUTE CONJUGATE MOMENTUM

$$\Pi = \frac{\partial\mathcal{L}}{\partial\phi_{,0}} = -\alpha\gamma^{\frac{1}{2}}\left(-\alpha^{-2}\phi_{,0} + \alpha^{-2}\beta^i\phi_{,i}\right)$$

$$= \frac{\gamma^{\frac{1}{2}}}{\alpha}\left(\phi_{,0} - \beta^i\phi_{,i}\right)$$

- SOLVE FOR $\phi_{,0} = \phi_{,0}(\Pi, \phi_{,i})$

$$\phi_{,0} = \frac{\alpha}{\gamma^{\frac{1}{2}}}\Pi + \beta^i\phi_{,i}$$

- COMPUTE HAM. DENSITY $\mathcal{H} = \Pi\phi_{,0} - \mathcal{L}$

$$\mathcal{H} = \Pi\phi_{,0} - \mathcal{L}$$

$$= \frac{\alpha}{\gamma^{\frac{1}{2}}} \pi^2 + \beta^i \phi_{,i} \pi$$

$$+ \frac{1}{2} \alpha \gamma^{\frac{1}{2}} \Big(-\alpha^{-2} \left(\frac{\alpha}{\gamma^{\frac{1}{2}}} \pi + \beta^i \phi_{,i} \right)^2 +$$

$$2\alpha^{-2} \left(\frac{\alpha}{\gamma^{\frac{1}{2}}} \pi + \beta^i \phi_{,i} \right) \beta^j \phi_{,j} +$$

$$(\gamma^{ij} - \alpha^{-2} \beta^i \beta^j) \phi_{,i} \phi_{,j} \Big)$$

- LAST TERM SIMPLIFIES ENORMOUSLY (EXERCISE)

$$\mathcal{H} = \frac{\alpha}{\gamma^{\frac{1}{2}}} \pi^2 + \beta^i \phi_{,i} \pi + \frac{1}{2} \alpha \gamma^{\frac{1}{2}} \left(-\gamma^{-1} \pi^2 + \gamma^{ij} \phi_{,i} \phi_{,j} \right)$$

$$\rightarrow \boxed{\mathcal{H} = \frac{1}{2} \frac{\alpha}{\gamma^{\frac{1}{2}}} \pi^2 + \beta^i \phi_{,i} \pi + \frac{1}{2} \alpha \gamma^{\frac{1}{2}} \gamma^{ij} \phi_{,i} \phi_{,j}}$$

- NOTE: IN MINKOWSKI S.T., IN A CICS:

$$\alpha = 1, \quad \beta^i = 0, \quad \gamma_{ij} = \gamma^{ij} = \mathrm{diag}(1, 1, 1)$$

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi$$

- DERIVE HAM. EOM FROM $\dot{\phi} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \phi}$

(a) VARY W.R.T. π

$$\delta H = \left(\frac{\alpha}{\gamma^{\frac{1}{2}}} \pi + \beta^i \phi_{,i} \right) \delta \pi \qquad \left(\int \text{ IMPLIED} \right)$$

$$\rightarrow \boxed{\dot{\phi} = \frac{\alpha}{\gamma^{\frac{1}{2}}} \pi + \beta^i \phi_{,i}}$$ AS PREVIOUSLY (EV 1)

(b) VARY W.R.T Φ

$$\delta H = \beta^i \Pi \, \delta\Phi_{,i} + \alpha \gamma^{\frac{1}{2}} \gamma^{ij} \Phi_{,j} \, \delta\Phi_{,i}$$

$$= -\left(\beta^i \Pi + \alpha \gamma^{\frac{1}{2}} \gamma^{ij} \Phi_{,j}\right)_{,i} \delta\Phi$$

$$\rightarrow \quad \boxed{\dot{\Pi} = \left(\alpha \gamma^{\frac{1}{2}} \gamma^{ij} \Phi_{,j}\right)_{,i} + \left(\beta^i \Pi\right)_{,i}} \qquad \text{(EV2)}$$

- CAN SHOW (EXERCISE) ~~THAT~~ (EV1), (EV2) ARE EQUIVALENT TO

$$\Box\Phi = 0 \quad \rightarrow \quad \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \, g^{\mu\nu} \Phi_{,\mu}\right)_{,\nu} = 0$$

- NATURAL HAMILTONIAN VBL'S MAY NOT BE IDEAL CANDIDATES FOR NUMERICAL WORK, PARTICULARLY IN CURVILINEAR COORDINATES — WILL GENERALLY PROVIDE GOOD "STARTING POINTS" HOWEVER