12.3. In our estimates of the length of a protein, we have used the end-to-end distance. This very simple (to calculate) measure may not tell the whole story. For example, a protein could contain a single fold or have a form like a "ball of string," and have the same end-to-end length. Investigate the behavior of the mean-square size calculated in the following way. Let $\vec{r}_{cm}$ be the location of the center of mass of the protein. One measure of the size of the chain is the quantity $\Delta \equiv < |\vec{r}_i - \vec{r}_{cm}|^2 >$ where $\vec{r}_i$ is the position of the ith amino acid and the angular brackets denote an average over all pieces of the chain. Calculate Δ as a function of temperature and compare its behavior to that of the end-to-end distance.

***12.4.** Perform a simulation of protein folding and examine the variation of the energy as a function of (Monte Carlo) time. Use this, and any other approaches you can devise, to reconstruct part of the energy landscape, as sketched in Figure 12.12. Can you say anything about the number of wells as a function of their depth? How does this distribution change as the protein chain is made longer?

***12.5.** Investigate the problem of metastability of protein folding by comparing the structure obtained by two or more separate simulations of the same model protein. Consider a chain with 30 amino acids, and let it find two or more metastable states by letting it fold at $T = 1$, as in Figure 12.11. Then compare the actual structures of the different folded states. Are the structures similar or very different? Can you estimate the size of the energy barriers that separate the different metastable states?

12.2 EARTHQUAKES AND SELF-ORGANIZED CRITICALITY

Earthquakes often have a large and dramatic impact, which makes them a topic of continuing interest. Earth's crust contains numerous fault lines that separate large pieces of material called *plates*. Each of these plates is fairly sturdy, but the connections across a fault line are relatively weak. Over time the crust deforms, exerting forces on the plates and leading to a gradual build up of potential energy. This energy is released by the sudden movement of one plate relative to an adjacent one. Such an event is an earthquake. While this general picture of earthquakes is well established, there are many questions that are not settled. For example, we would like to know how to predict when earthquakes will occur and how large the next quake associated with a particular fault line will be.

Geoscientists have been involved in modeling earthquakes for many years. In this section we follow their lead and model two adjacent pieces of Earth's crust as masses that are able to slip past each other in response to a steadily increasing force. Such a mechanical model involves Newton's second law, which gives a small excuse for considering this to be a physics problem. However, there is another feature of earthquakes that makes them of interest to physicists. It has been proposed that earthquakes may have some important features in common with the second order phase transition we observed in connection with the Ising model in Chapter 8. In order to understand this connection we need to introduce the so-called Gutenberg-Richter law, which can be stated as follows. The size of an earthquake is often measured using the Richter scale, which is commonly referenced by the popular press. This is a logarithmic scale involving the magnitude of an earthquake. The amount that one of Earth's plates shifts relative to another during an earthquake is proportional to the moment of the event, M. This quantity is also proportional to the energy released by the event. The magnitude of the quake, $\mathcal{M}$, is equal to the logarithm of the moment, so an earthquake with a magnitude of 7 on the Richter scale is much more powerful than an event whose magnitude is 6.

A logarithmic scale is convenient because earthquakes come in an extremely wide range of sizes. Fortunately, the number of large quakes is much smaller than the number of little ones. This "preference" for small events is well-documented from observations and also follows a logarithmic form. This is known as the Gutenberg-Richter law and can be stated mathematically as

$$P(\mathcal{M}) = A\,M^{-b} = A\,e^{-b\mathcal{M}} . \tag{12.2}$$

Here $\mathcal{M} \equiv \ln M$ is the magnitude of an event,[12] $P(\mathcal{M})$ is the probability (per unit $\mathcal{M}$) of having a quake of a given magnitude, A is a constant, and b is a factor that lies somewhere in the range 0.8–1.5. The use of the term law in connection with (12.2) is perhaps a bit too strong, as it is really just an empirical rule that has been found to describe the distribution of earthquake magnitudes observed for many different fault lines. Surprisingly, there is no fundamental understanding of why earthquakes (or Earth itself?) follow this rule. In particular, why doesn't $P(\mathcal{M})$ vary as $e^{+\mathcal{M}}$, or even $\mathcal{M}^{-\pi}$?

The Gutenberg-Richter law is also interesting for what it implies about the amount of energy released in a typical event. Since the energy released is proportional to M, the average energy of an event is just the integral of M over the distribution (12.2)

$$E_{\text{average}} = \int_0^\infty E\,A\,e^{-b\mathcal{M}}\,d\mathcal{M} \sim \int_0^\infty M\,e^{-b\mathcal{M}}\,d\mathcal{M} . \tag{12.3}$$

Since $M \sim e^{\mathcal{M}}$ (and given the observed range of b) this integral *diverges*! Fortunately it appears that such "average" earthquakes don't happen very often.[13] More seriously, power law distributions such as (12.2) which have awkward (or infinite) averages, are quite rare in nature. Perhaps the best-documented and understood case in which such distributions occur is near a second-order phase transition. You may recall that in our studies of the Ising model in Chapter 8 we noted that many quantities exhibit power law singularities at a critical point. For example, the correlation length associated with fluctuations of the magnetization is infinitely large when $T = T_c$. This analogy has suggested to some researchers (see Bak and Tang [1989]) that Earth is effectively located at a critical point as far as earthquakes are concerned. The fact that there is a sort of earthquake phase transition for a certain value of Earth's density and temperature, etc., might seem unlikely, but it is at least plausible. However, it would be even more surprising to find that Earth just happens to have a temperature and density that place it very near this transition. The interesting speculation is that some feature of this phenomena *automatically* causes Earth to be located at the transition. This scheme is known

[12]We could choose to use either natural or base-10 logarithms. Here we follow the convention employed in Carlson and Langer (1989) and Carlson (1991).

[13]This strongly suggests that the Gutenberg-Richter law must break down at large $\mathcal{M}$.

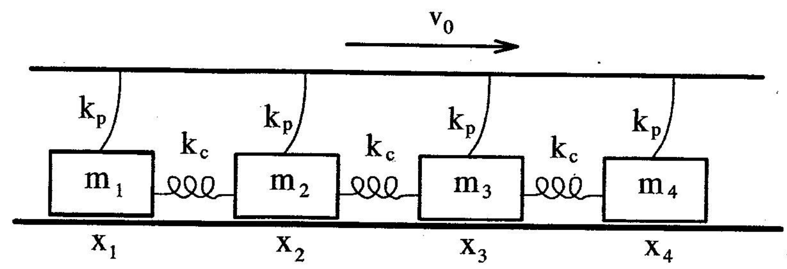

FIGURE 12.14: Model of two plates separated by a fault line at which an earthquake can occur. Imagine this to be a top view of Earth's surface, with the fault line running between the blocks and the bottom plate.

as *self-organized criticality.* The relevance of this concept to any real systems is not universally accepted, but this discussion does emphasize the striking nature of power law distributions such as (12.2). It has also prompted many researchers to try to account for such distributions in terms of (semi)realistic models, and that is the subject of this section.

The model we consider is copied from one proposed by Burridge and Knopoff (1967), and discussed by Carlson and Langer (1989); it is shown in Figure 12.14. We imagine that two of Earth's plates are moving slowly relative to one another. One of the plates is the bottom surface (the lower, thick horizontal line) in Figure 12.14, while the other plate is the top surface. Caught between them is a portion of the crust modeled by a collection of blocks. For simplicity we will assume that the blocks are arranged in a line, but we can also consider a two-dimensional array (we will explore this possibility in the exercises). The blocks are connected to each other by a force that is modeled as springs, with force constants k_c. The blocks are also connected to the top plate via "leaf" springs,[14] k_p. The only other force in the problem is a frictional force between the blocks and the bottom plate, which we will describe in detail shortly.

The top plate in Figure 12.14 is assumed to move to the right with a constant velocity v_0. Thus, through the leaf springs it exerts an ever-increasing force on the blocks. When this force is small, the frictional force from the bottom plate will prevent the blocks from moving, and energy will build up in the potential energy of the leaf springs. Eventually the force from these springs will overcome the frictional force, and one or more blocks will move suddenly. This is an earthquake. Since the blocks are connected to each other by the springs k_c, the motion of one block can cause other blocks to move as well. If this motion spreads to involve many blocks, one slip will lead to a large quake.

An important ingredient in the model is the frictional force between a block and the bottom surface. We refer to this force as friction because it is assumed to exhibit the general features we all learn in our elementary mechanics courses. When a block is stationary relative to the bottom plate the force is static friction, while if

[14] While most springs have a helical form, leaf springs are a single strip of material. They resist bending (that is, "spring" back) much like a stem or leaf would if they were bent and then released.

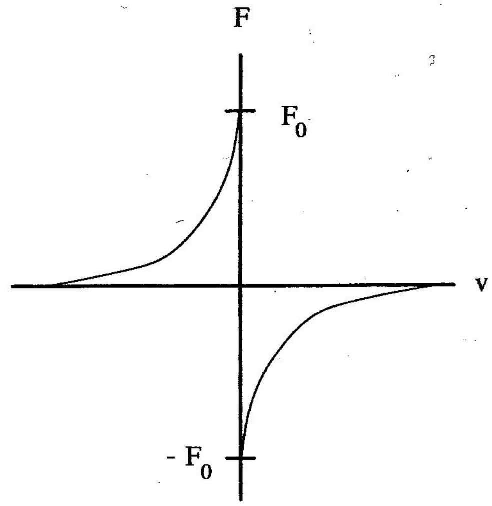

FIGURE 12.15: Schematic of the frictional force between a block and the bottom plate as a function of the velocity of the block.

it is moving we are dealing with kinetic friction. Furthermore, we are taught that the maximum force of static friction is greater than the force of kinetic friction. This is sometimes referred to as a stick-slip force, since once an object begins to move, the frictional force becomes smaller, resulting in a sudden increase in the velocity. Figure 12.15 shows the general form we will use for the frictional force. It always opposes the relative motion of the block and the surface, and is largest when the velocity of a block relative to the bottom plate, v, vanishes. We will follow previous work on this model (see Carlson and Langer [1989]), and assume that the magnitude of this frictional force decreases with increasing $|v|$. Unfortunately, there is no fundamental understanding of such frictional forces (in contrast, for example, to the Van der Waals force), so it is hard to put this important feature of the model on a solid basis. We can, however, investigate how the form of this force affects the properties of the model, and we will return to this point below.

Before we consider in detail how to calculate the behavior of the model, it is worthwhile to make a few comments with regard to model building. Our goal in studying this model is *not* to reproduce the detailed behavior of Earth near any particular fault line. Instead, our aim is to determine what properties a system must have to exhibit a power law distribution of earthquake sizes (12.2). We have already encountered systems with simple harmonic forces similar to the spring forces in Figure 12.14.[15] The only unusual feature of the model we are considering here is the frictional force, so we expect that this must be the key for obtaining power law behavior. Confirming this suspicion, or showing it to be false, will be our initial concern. If we do find power law behavior, then we can conclude that we have

[15]We saw in Chapter 3 that this force leads to simple harmonic motion. The interesting chaotic behavior was found only when we considered deviations from a purely harmonic force.

(perhaps) captured the essential physics of the problem. If power law behavior is not found, we will be forced to continue our search to identify the key ingredient responsible for (12.2). This could require that we consider other functional forms for the frictional force, or that we generalize the model in other ways.

This is the motivation behind model building in theoretical physics. The goal is not simply to construct a simulation that reproduces nature,[16] but rather to identify the essential physics responsible for the interesting behavior. This process can, of course, be iterated as we attempt to bring a model ever closer to reality.

Now let us return to our earthquake model and consider the forces in a little more detail. The model consists of N blocks whose positions are x_i where i ranges from 1 to N (see Figure 12.14), and for simplicity we assume that all have the same mass, m. The force between two adjacent blocks is due to the spring that connects them. The force from a spring is given by Hooke's law, and has the form $F = -k\Delta x$, where k is the force constant and Δx is the amount that the spring is stretched or compressed relative to its relaxed state. It is convenient to measure each x_i with respect to the equilibrium position of block i. Hence, $x_i = 0$ if a block is at its equilibrium location. The force on block i from its neighboring blocks is then

$$F_b = -k_c(x_i - x_{i+1}) - k_c(x_i - x_{i-1}) . \qquad (12.4)$$

Finally, we note that our system will have "free" ends. The blocks at each end will be connected to only one other block.[17]

The force of the leaf spring on block i has a similar form, $F = -k_p(x_i - x_{\text{leaf}})$. We assume that at $t = 0$ the leaf springs are all unstretched, so that initially $x_{\text{leaf}} = 0$ for each block. The horizontal bar moves with velocity v_0, so x_{leaf} increases with time according to $x_{\text{leaf}} = v_0 t$. The force of the leaf spring on block i is then

$$F_l = -k_p(x_i - x_{\text{leaf}}) = -k_p(x_i - v_0 t) . \qquad (12.5)$$

The only remaining force is that due to friction with the bottom plate. We will assume that it has the form shown in Figure 12.15. When the velocity of a block is zero the frictional force will take on whatever value is necessary to keep the block at rest. That is, the frictional force will oppose the other forces on the block so that the sum of all of the forces (friction included) vanishes. However, the static frictional force is limited to a maximum magnitude of F_0, so if the sum of the other forces exceeds this level, the block will experience a nonzero force and begin to move. If the block is moving we are then dealing with kinetic friction, which we will assume is given by

$$F_f = -\frac{F_0 \,\text{sign}(v_i)}{1 + |v_i/v_f|} , \qquad (12.6)$$

[16] In such a case we could say that the *computer* understands the problem; we want to understand it, too.

[17] We will leave the study of the effects of periodic boundary conditions to the interested reader. In this problem periodic boundary conditions seem unphysical. In particular, we might imagine that some earthquakes start at the end of a fault and propagate inward. Such behavior would not be possible if the model employed periodic boundary conditions.

where v_f is a parameter that determines the velocity dependence of the force. When $v_i = v_f$, the frictional force drops to half of its $v_i = 0$ value. The factor $\mathrm{sign}(v_i)$ ensures that F_f always opposes the motion.

Using springs to model the interactions between blocks and between a block and the opposite side of the fault line may seem a bit contrived, but it is actually on firm mathematical footing for the following reason. The energy of interaction between two blocks will, in general, be a function of the separation between the blocks; let us call this function $U(\Delta x)$, where $\Delta x \equiv x_{i+1} - x_i$. Assuming that $U(\Delta x)$ is a well-behaved function, we can perform a Taylor expansion

$$U(\Delta x) \;=\; U(0) \,+\, (\Delta x)\,U' \,+\, \frac{(\Delta x)^2}{2}\,U'' + \ldots, \qquad (12.7)$$

where U' is the first derivative of U evaluated at $\Delta x = 0$, etc. The corresponding force is $F = -dU/d(\Delta x)$

$$F(\Delta x) \;=\; -\,U' \,-\, (\Delta x)\,U'' \,- \ldots. \qquad (12.8)$$

By definition, this force vanishes when the blocks are at their equilibrium spacing, so U' must be zero. For small Δx we thus have $F \approx -(\Delta x)U''$, which is just Hooke's law with $k = U''$. Hence, the form of Hooke's law is a natural result for a force that arises from a well-behaved (Taylor expandable) potential energy function. This is one reason why springs are a popular ingredient in the models devised by physicists. They are in fact a very natural and general way to describe an interaction.

On the other hand, the basis of the frictional force (12.6) is not nearly as firm. As we have already noted, there is no fundamental understanding of friction. The best we can do is assume a simple form such as (12.6) and study the kind of behavior it yields. We will return to this point later.

Putting all of these forces together with Newton's second law yields an equation of motion for each block

$$m_i\,\frac{d^2x_i}{dt^2} \;=\; k_c\,(x_{i+1} + x_{i-1} - 2x_i) \,+\, k_p\,(v_0 t - x_i) \,+\, F_f\,. \qquad (12.9)$$

This can be written as two first-order differential equations,

$$\frac{dx_i}{dt} \;=\; v_i\,, \qquad (12.10)$$

$$m_i\,\frac{dv_i}{dt} \;=\; k_c\,(x_{i+1} + x_{i-1} - 2x_i) \,+\, k_p\,(v_0 t - x_i) \,+\, F_f\,, \qquad (12.11)$$

and this system of equations can be solved using the Euler method. As usual, we discretize time into steps Δt. At every time step we use the velocity of each block to estimate its position at the next step. We also calculate the force on each block and use it to obtain the velocity at the next time step. Note that the forces are functions of the current positions, so to be consistent with respect to the spirit of the Euler method we must calculate the forces on *all* of the blocks before updating the positions and velocities.[18]

[18] Updating in a different order could easily yield the Euler-Cromer method. For this problem the Euler and Euler-Cromer methods are both acceptable algorithms.

The programming is similar to what we have encountered in several previous cases, including the projectile and pendulum problems. For a system of N blocks we have to keep track of N different positions along with the corresponding velocities. The only really new feature is the frictional force. In order to model this force properly there are several different cases that must be considered.

- The block is not moving at time-step n, and the sum of the forces from the block springs and the leaf spring is smaller (in magnitude) than F_0. The static frictional force will then adjust itself to precisely cancel the other forces. Since the total force will thus be zero, the velocity at time step $n + 1$ will also be zero.
- The block is not moving at step n, and the sum of the forces from the block springs and the leaf spring is greater than F_0. The frictional force will have a magnitude of F_0, and oppose the sum of the other forces. The total force will not be zero and the velocity at the next time step will be nonzero.
- The block is moving at step n. We calculate the velocity for time step $n+1$ using the (kinetic) frictional force (12.6), along with the forces from the block and leaf springs. If this new velocity has the same sign as the velocity at step n, everything is fine and the calculation proceeds in the usual way. However, if the new velocity would be *opposite* to the previous velocity, this means that the frictional force is sufficiently large that it will "capture" the block; that is, bring it to rest. In this case the velocity at time step $n + 1$ must be set to zero.

This description of the frictional force is actually much longer than the number of lines needed to implement it in a program. However, it does show that a force that is a discontinuous function generally requires some extra care.

We are now ready to consider the behavior of our earthquake model. It contains 5 parameters, k_p, k_c, m, F_0, and v_0. Some of these can be effectively removed by the appropriate choice of units, but there will still be a large number of parameter choices to explore. In most of the simulations below we will use the following values: $m = 1$, $k_p = 40$, $k_c = 250$, $F_0 = 50$, and $v_0 = 0.01$. These values appear to give fairly typical behavior; other parameter values will be considered in the exercises (see also Carlson and Langer [1989] and Carlson [1991]).

In addition to these parameters, we must specify the initial conditions. For simplicity we will always assume that the initial velocity of each block is zero, but this still leaves us with the choice of initial positions. One choice is to begin with the blocks all located in their equilibrium positions. In this case the forces on all of the blocks from both types of springs is zero at $t = 0$. Such a perfectly ordered start is not very realistic, but is useful for illustrating a few important points. Some results for this case are shown in Figure 12.16, where we plot the position and velocity of a particular block as functions of time. This simulation involved 25 blocks, but since the initial conditions were uniform, the behavior of every block was the same as that shown here. That is, they all moved together, and the springs k_c were never stretched or compressed. We see from Figure 12.16 that the block remained at $x = 0$ until $t \sim 120$. During this time the opposite side of the fault was moving steadily at speed v_0 and the force from the leaf spring gradually increased. This force was not able to overcome friction until $t = 120$, at which point the *entire* system of blocks began to move. The blocks moved approximately 2 units

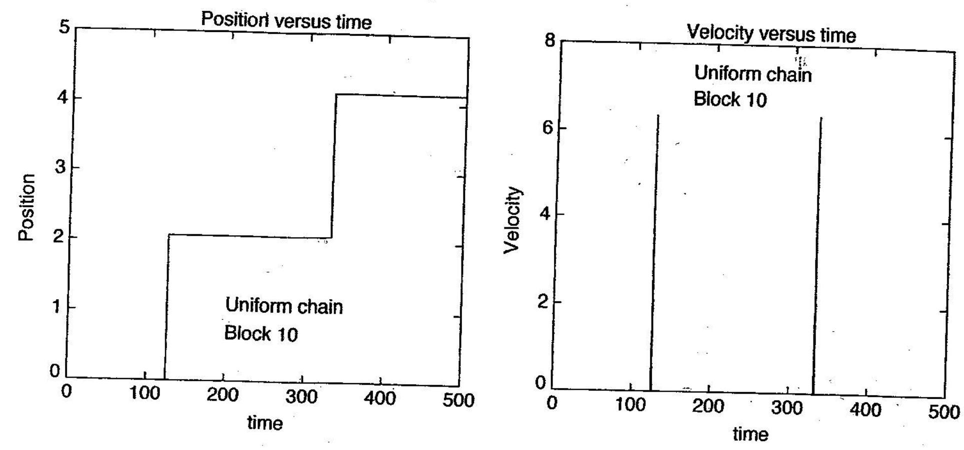

FIGURE 12.16: Behavior of block 10 in a 25-block system, with a completely ordered start. Left: position of block 10 as a function of time; right: velocity of the same block. The velocity was zero except for the two very narrow spikes at $t \sim 120$ and ~ 330.

before the frictional force brought them to a halt. The process then repeated. The force from the leaf springs grew until it overcame friction at $t \sim 330$, and the blocks all moved again, etc. The corresponding velocity is also shown. It was zero until the blocks moved, at which point it exhibited a large but very narrow peak. Of course, this is just the derivative of the position as a function of time.

The two displacements in Figure 12.16 correspond to an abrupt motion of the system of blocks. These are earthquakes. The quakes found with an ordered start are very special, since the blocks all move together and the events occur at regularly spaced intervals.[19] This will change when we start the blocks with random initial positions. However, before we do that it is useful to compare the behavior in Figure 12.16 with some simple, analytic results. Because of the special initial conditions, the block springs k_c were never stretched or compressed. We thus need consider only a single block moving in response to the leaf spring and the force of friction. Initially, the force from the leaf spring was $k_p(v_0t - x) = k_pv_0t$, since $x = 0$ was the initial condition. The block will not move until this exceeds F_0, which occurs at $t = F_0/(k_pv_0)$. For the parameters used in here this yields $t = 120$, in good agreement with Figure 12.16. The block should then move until the frictional force (12.6) is equal to the force from the leaf spring. We will leave it to the exercises to check that the displacements in the two events in Figure 12.16 agree with the expected value. The peaks in the velocity can be estimated in a similar way.

The results in Figure 12.16 are useful since they allow us to check our program against analytic results. This behavior also brings out an important programming issue. The earthquakes occur over a very short time compared to the interval

[19] We only showed two quakes here, but if we had shown the behavior for longer times, you would have seen that the quakes do indeed repeat at regular intervals.

between quakes. The duration of a quake is too small to resolve on the scale in Figure 12.16, but is of order 0.5 time units. This implies that the time step in our simulation must be a small fraction of this to avoid significant numerical errors. Our usual practice is to use a time step that is 1% of the characteristic time scale of the problem, which would thus be ~ 0.005. The time between quakes is on the order of 100 time units, so this would lead to $\sim 2 \times 10^4$ time steps between events. A simulation using a large number of blocks would thus take a lot of computer time. Moreover, there would be nothing happening for the vast majority of this time.

There are two ways to deal with this problem: use a fast computer and just be patient, or use *two* different time steps according to the following strategy.[20] During the times when no blocks are moving we use a (relatively) large time step. The only motion during these periods involves the top plate, and since it moves with a constant velocity, the use of a large time step does not introduce any errors. A small time step (~ 0.005 in the above example) is used during the times when the blocks are moving. This strategy is straightforward, except that we need to have a systematic way to switch back and forth between time steps. One convenient way to make these switches is as follows. When the blocks are not moving, the larger time step is used to calculate the new velocities. If the velocity of *any* block is nonzero at the next time step, an earthquake is imminent. We then "back up" to the previous time step and continue the calculation with the smaller value of the time step until after the upcoming quake is finished. When the velocities of all of the blocks are again zero, the time step is set back to the larger value and the calculation proceeds. The results in Figure 12.16 were obtained with this algorithm, using a large time step of 0.03 and a small time step of 0.003. We will leave it to the exercises to check that these values are sufficiently small that the numerical errors were negligible.

A simulation with an initially ordered configuration is not very realistic, since we don't expect Earth's crust to ever be perfectly uniform. The behavior is quite different when the blocks are given a disordered initial configuration. If we displace them initially from their equilibrium positions by random amounts in the range ± 0.001, we find the results shown in Figure 12.17. Here we show only four quakes and it is seen that they had different magnitudes, that is, different total displacements. In addition, the time until the next quake varied from event to event.

It is intriguing that such a small initial displacement (only 0.1% of the spacing between blocks) is able to produce such dramatically different behavior. It turns out that the behavior found in Figure 12.16, with a perfectly ordered start, is *unstable* with respect to any initial displacements from equilibrium, no matter how small. That is, the behavior is extremely sensitive to small deviations from a perfectly ordered start. This should remind you of chaotic systems and their extreme sensitivity to initial conditions; this is our first indication that this earthquake model is not a "simple" mechanical system.

[20]The use of two (or more) time steps is often referred to as an "adaptive" step-size procedure. The example we describe here is a very simple one, but does illustrate the basic idea. Such an approach is useful in simulations involving functions or behaviors that have significant structure limited to small regions of time or space.

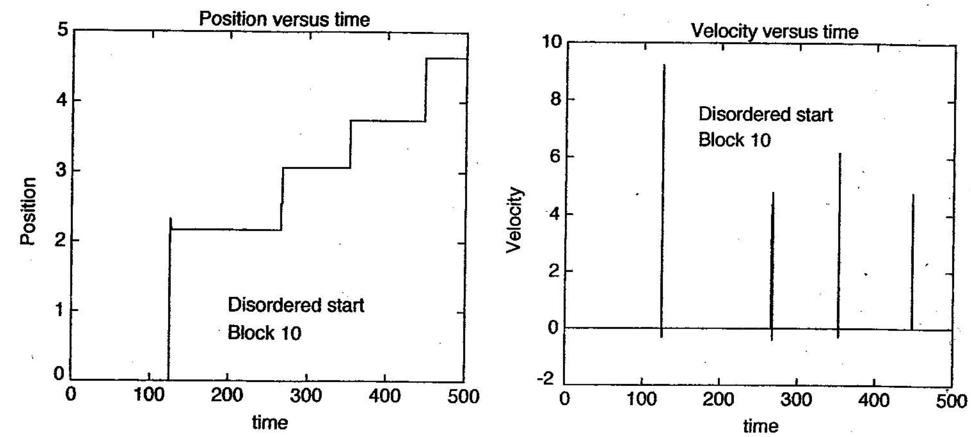

FIGURE 12.17: Behavior of block 10 in a 25-block system. Each block was given a random initial displacement from its equilibrium position. This displacement was in the range -0.001 to $+0.001$. Left: position of block 10 as a function of time; right: velocity of the same block.

So far we have examined the behavior by following the motion of a single block. However, since we expect that there can be earthquakes that do not involve all of the blocks, it is useful to view the same behavior using the perspective plots in Figures 12.18 and 12.19, which show the behavior of the entire system. With an ordered start all blocks move together, as we had anticipated. In contrast, with a disordered start the quakes are much less organized. There are numerous events in a time interval that would contain only one or two quakes for the case of an ordered start. Some of these events involve many blocks, while in others only a few blocks are in motion. We thus have a distribution of earthquake sizes.

One of our primary goals is to try to understand the origin of the Gutenberg-Richter law, and to do this we need to add one more feature to the simulation. As we mentioned earlier, the magnitude of an earthquake is the natural logarithm of the earthquake moment. The moment M is proportional to the total displacement, which can be found by summing (integrating) $v_i \Delta t$ for each block over the course of the event. The moment of an event is thus

$$M = \sum_{n=\text{time}} \left(\sum_{i=\text{blocks}} v_i \Delta t \right) , \tag{12.12}$$

where the sums are over all blocks i and over the time steps n for which the velocities are not all zero. The magnitude of the event is then $\mathcal{M} \equiv \ln M$. After accumulating the results for a large number of events we can obtain the distribution $P(\mathcal{M})$ by dividing the $\mathcal{M}$ axis into bins and counting the number of events that fall into each bin. Results for $P(\mathcal{M})$ are shown in Figure 12.20, where the figure on the left shows the distribution for the system of 25 blocks we have considered in all cases to this point. On this semilogarithmic plot the Gutenberg-Richter law (12.2) is a straight line with slope $-b \log_{10}(e) \approx -0.43b$. The results from the simulation are certainly

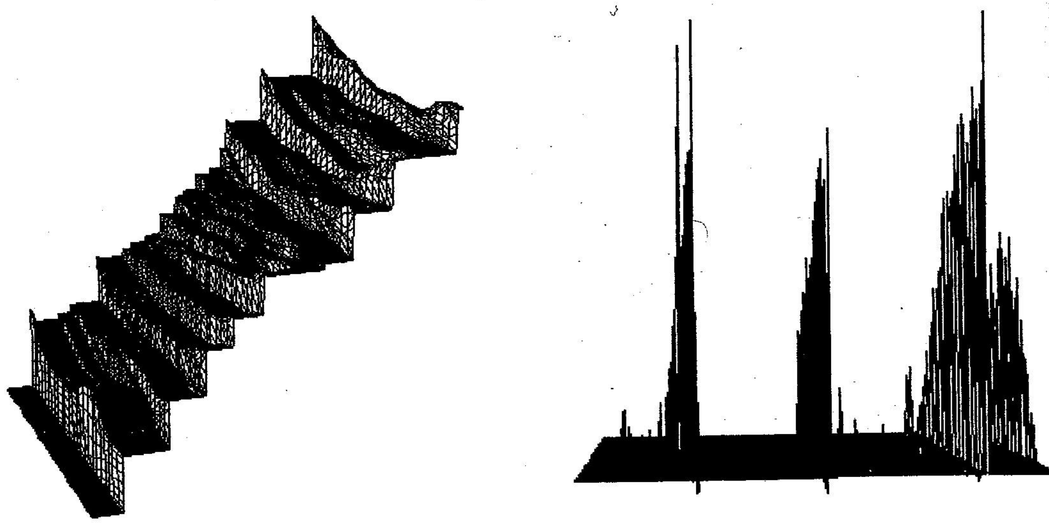

FIGURE 12.19: Results for the simulation of Figure 12.17 in which the blocks were in an initially disordered configuration. Time goes from left to right, block number from front to back, and the vertical axis is position (left figure) or velocity (right figure). The time span covered in the position plot is $t = 0$ to 1000, but for purposes of clarity the velocity plot shows a smaller range, $t = 300$ to ~ 500.

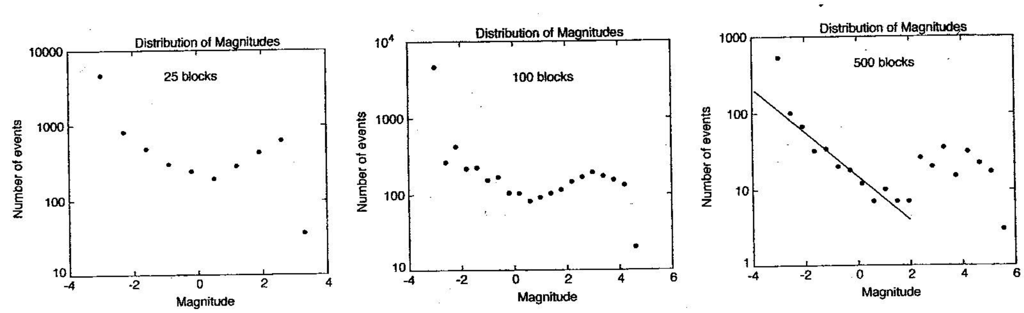

FIGURE 12.20: Earthquake distributions for systems of various sizes. These histograms were obtained from a collection of approximately 7000 events for the 25- and 100-block systems and 3000 events for the 500-block simulation. The vertical axis is the number of events per histogram bin and is thus proportional to the probability density of events.

in explaining the Gutenberg-Richter law. However, we have been able to show that this simple mechanical model is capable of exhibiting power law behavior over at least a limited range. To some small extent this lends support to the proposals concerning self-organized criticality mentioned at the beginning of this section. As for the relevance to real earthquakes, it has been suggested that the Gutenberg-Richter law may actually fail at large $\mathcal{M}$, so perhaps part of our problem is with the law itself, rather than the model.[22] Of course, it is also possible that the problem lies with the model. You will recall our philosophy of model building, according to which we strive to construct the simplest model that contains the essential physics of the phenomena of interest. It is certainly conceivable that our simple model has omitted some key element(s). Possibilities include the following: (1) The dimensionality of the fault system; a two-dimensional array of masses might be more appropriate than the one-dimensional arrangement considered above. Here the second spatial dimension would correspond to depth beneath Earth's surface. (2) We have assumed uniform values of m, k_c, and k_p. For a real fault the analogous parameters will not be constants, but vary with position. (3) The frictional law (12.6) has no fundamental basis. We could certainly imagine other plausible possibilities. These are just a few of the ways in which the model could be modified, and we will leave such studies to the exercises. While we have not been able to answer all of the questions concerning earthquakes posed at the beginning of this section, these simulations do shed some light on the problem, and serve to illustrate the model-building process in theoretical physics.

EXERCISES

12.6. Consider the simulation in Figure 12.16 in which 25 blocks were given a perfectly ordered arrangement at the start. Continue this simulation to longer times and show that the earthquakes occur at regularly spaced intervals (as we claimed above).

12.7. Perform a simulation with 25 blocks, allowing for some randomness in either the masses (let m_i vary from 0.5 to 2.0) or in the spring constants (either k_c or k_p). Compare your results for the distribution of earthquake magnitudes with the results in Figure 12.20. The objective is to see if adding some disorder can lead to better agreement with the Gutenberg-Richter law or reduce the excess number of events at high $\mathcal{M}$, or both.

12.8. Assume that the blocks are all initially in their equilibrium positions and obtain analytic estimates for the time between quakes, the displacement of a block during a quake, and the maximum velocity during a quake. Compare these estimates with the results in Figure 12.16.

***12.9.** Explore the properties of a two-dimensional earthquake model. A calculation of this kind is described in the references.

***12.10.** Investigate how the distribution of earthquake magnitudes depends on the form chosen for the frictional force. As an example, consider the case $F_f = F_0$ when $v = 0$ (static friction) and $F_f = -\text{sign}(v)F_0/2$ for $v \neq 0$ (kinetic friction). You should find (see Figure 12.21) that with this friction law there is no longer an excess of events at large $\mathcal{M}$, so the results are more realistic than that obtained

[22] This is a difficult issue to resolve, since the number of large quakes is (fortunately) small, making it hard to get a good estimate of $P(\mathcal{M})$ at large $\mathcal{M}$.

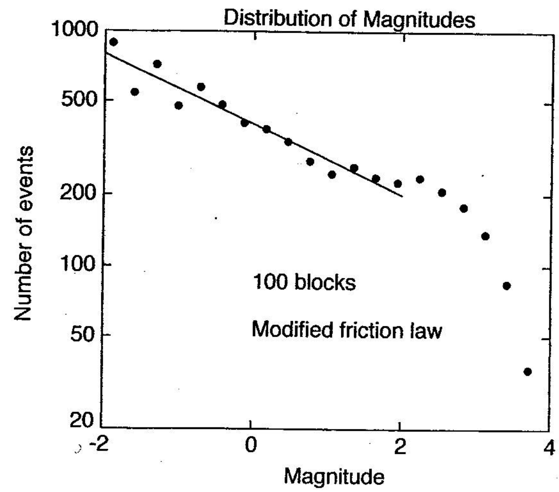

FIGURE 12.21: Earthquake distribution for a system of 100 blocks and the friction law described in Exercise 5. The straight line corresponds to the Gutenberg-Richter law (12.2) with $b \sim 0.35$.

with the frictional force (12.6). However, the slope for small $\mathcal{M}$ now corresponds to a value of b that is much smaller than 1, so the model still seems to lack an important ingredient. Study the behavior with other forms for the frictional force and try to determine one that gives a power law with a larger value of b.

12.3 NEURAL NETWORKS AND THE BRAIN

The Ising model consists of a large number of very simple units, that is, spins, which are connected together in a very simple manner. By "connected" we mean that the orientation of any given spin s_i, is influenced by the direction of other spins through the interaction energy Js_is_j. The behavior of an isolated spin, as outlined in our discussions leading up to mean-field theory, was unremarkable. Things only became really interesting when we considered the behavior of a large number of spins and allowed them to interact. In that case we found that under the appropriate conditions some remarkable things could occur, including the singular behavior associated with a phase transition. In this section we will explore a rather different system, which shares some of these features.

The human brain consists of an extremely large number ($\sim 10^{12}$) of basic units called neurons, each of which is connected to many other neurons in a relatively simple manner. A biologically complete discussion of neurons and how they function is a long story. Here we will give only a brief description of those features that seem to be most relevant to a physicist's understanding of the brain.A schematic picture of two neurons is given in Figure 12.22. Each neuron has a body (called a soma), along with dendrites and an axon.[23] The size scale depends on the type of neuron,

[23]There are other parts as well, but keep in mind that we are giving only a simplified description here.