Outer boundary conditions in General Relativity

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Outline

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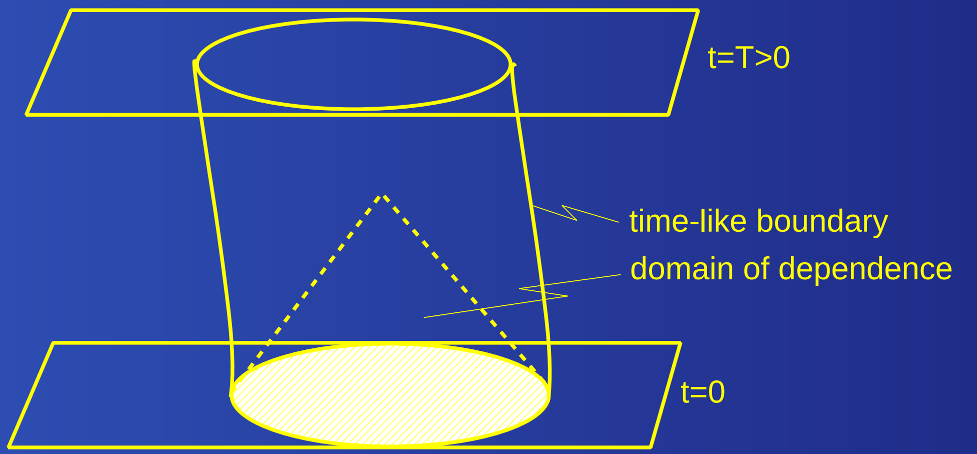
- Introduction
- Formulation
- Constraint-preserving boundary conditions
- Determinant condition
- Numerical results
- A related toy model problem in ED

Introduction

Solve Einstein's equations in a domain with timelike boundaries.



Boundary conditions should

(i) be compatible with the constraints (constraint-preserving)

(ii) be physically reasonable (e.g. minimize reflections)

(iii) yield a well posed initial-boundary value formulation

Introduction

- A well posed initial-boundary value formulation was given by Friedrich & Nagy, 1999 in terms of a tetrad-based Einstein-Bianchi formulation.
- Numerical implementation for related formulation is underway (Reula, Bardeen, Buchman, S,...?)
- Less is known for metric-based formulations (although recent progress by Cornell-Caltech group and S & Tiglio).
- Relevant for: Outer/interface boundary conditions; constraint projection, elliptic gauge conditions,...

Formulation

Evolution equations can be cast into first order quasilinear form:
Frittelli & Reula, Anderson & York, Hern, KST,..., S & Tiglio

$$
\begin{aligned}
\pounds_n \alpha &= -\alpha K, \\
\pounds_n g_{ij} &= -2K_{ij} \ , \\
\pounds_n K_{ij} &= \frac{1}{2} g^{ab} \left(-\partial_a d_{bij} + 2\partial_{(i} d_{|ab|j)} - \partial_{(i} d_{j)ab} - 2\partial_{(i} A_{j)} \right) + \gamma \, g_{ij} H + \text{l.o} \\
\pounds_n d_{kij} &= -2\partial_k K_{ij} + \eta \, g_{k(i} M_{j)} + \chi \, g_{ij} M_k + \text{l.o.} \\
\pounds_n A_i &= -K A_i - g^{ab} \partial_i K_{ab} + \xi M_i + \text{l.o.}
\end{aligned}
$$

with some parameters γ, η, χ ξ.
Constraints: $H = 0$, $M_j = 0$ (Hamiltonian and momentum),
$d_{kij} = \partial_k g_{ij}$, $A_i = \partial_i \alpha / \alpha$.

Formulation

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Main evolution system has the form

$$\partial_t u = P^i(u)\partial_i u + F(u),$$

where $u = (\alpha, g_{ij}, K_{ij}, d_{kij}, A_k)$.
The constraint variables $v = (H, M_j, d_{kij} - \partial_k g_{ij}, A_i - \partial_i \alpha/\alpha, ...)$ satisfy the constraint propagation system

$$\partial_t v = Q^i(u)\partial_i v + B[u]v,$$

Provided that the parameters γ, η, χ ξ satisfy suitable inequalities, these two systems can be brought into *strongly hyperbolic form*.
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$\partial_t u = P^i(u)\partial_i u + F(u)$ is called *strongly hyperbolic* if there exists $K > 0$ and a symmetric matrix-valued function $H(u,n)$ which is smooth in u and n such that $K^{-1} \leq H(u,n) \leq K$ and $H(u,n)P^i(u)n_i$ is symmetric for all $n \in S^2$ and all u.

Constraint-preserving b.c.

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Solve equations on domain Ω with (smooth) boundary $\partial\Omega$.

- Start with the constraint propagation system,

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- Specify maximal dissipative boundary conditions:

$$E(t) \equiv \int_\Omega v^T H v \; d^3x, \qquad \frac{d}{dt}E(t) \leq \int_{\partial\Omega} v^T HQ(n)v \, dS + \frac{1}{\tau}E(t).$$

$v^T HQ(n)v = v_{in}^T \Lambda_+ v_{in} - v_{out}^T \Lambda_- v_{out}$. Set $v_{in} = 0$ (3 b.c.).
In this case we have an energy estimate $E(t) \leq e^{t/\tau}E(0)$.
In particular, this implies that $v(t) = 0$ if $v(0) = 0$.

Constraint-preserving b.c.

- Go back to main evolution system, $\partial_t u = P^i(u)\partial_i u + F(u)$.

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- Go back to main evolution system, $\partial_t u = P^i(u)\partial_i u + F(u)$.
- Boundary matrix $H(u,n)P^i(u)n_i$ has six positive eigenvalues; for high-frequency plane waves propagating towards the boundary:
  three constraint-violating modes; fields $u_{in}^{(cons)}$
  two physical modes; fields $u_{in}^{(phys)}$
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- We can set $u_{in}^{(phys)} = h$, where h is some a priori given boundary data.
- Set $u_{in}^{(gauge)} = 0$.

Constraint-preserving b.c.

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A different way of specifying boundary data is through the Weyl scalars Ψ_0 and Ψ_4, constructed from an adapted NP tetrad at the boundary:

$$\Psi_0 = c\Psi_4^* + h.$$

where $|c| < 1$.

Notice:

For linear fluctuations about a Schwarzschild black holes and spherically symmetric outer boundary, Ψ_0 and Ψ_4 are gauge-invariant quantities.

Determinant condition

Consider linear hyperbolic system with constant coefficients (high-frequency limit),

$$\partial_t u = \mathcal{A}u,\ t > 0,\ x > 0,$$

where $\mathcal{A}u \equiv A^x \partial_x u + A^y \partial_y u + A^z \partial_z u$ with differential boundary conditions

$$M(\partial_x, \partial_y, \partial_z)u = h(t, y, z).$$

Look for solutions of the form $u(t, x, y, z) = e^{st + i(w_y y + w_z z)} f(x)$, where $Re(s) > 0$, w_y, w_z real.

Test: If $h = 0$ there should be no such solutions. Otherwise the system is ill posed: Because if there is such a solution for some s, $Re(s) > 0$, then there is also a solution u_α for αs, $\alpha > 0$ and for each fixed t

$$|u_\alpha(t, x, y, z)|/|u_\alpha(0, x, y, z)| = e^{\alpha Re(s) t} \longrightarrow \infty.$$

(i.e. the operator $s - \mathcal{A}$ is not invertible for all $Re(s) > 0$.)

Determinant condition

Introducing the ansatz $u(t,x,y,z) = e^{st+i(w_y y + w_z z)} f(x)$ into the evolution and boundary equations gives

$$sf = A^x \partial_x f + i(A^y w_y + A^z w_z) f, \qquad L(s, iw_y, iw_z) f = 0.$$

Solution has the form $f(x) = P e^{M_- x} \sigma_-$, $Re(M_-) < 0$ with $L P \sigma_- = 0$.
Therefore, one has to verify the determinant condition

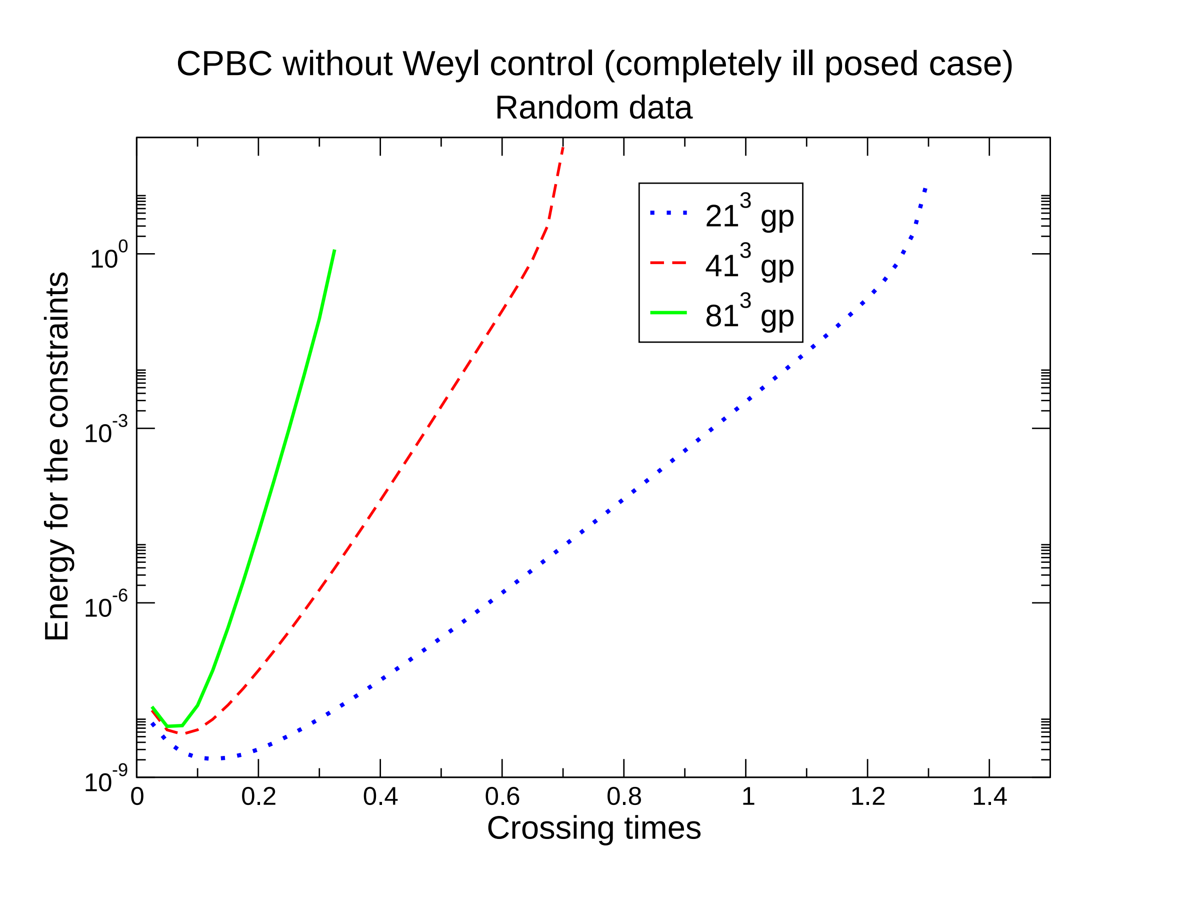
$$\det(LP)(s, w_y, w_z) \neq 0, \qquad Re(s) > 0.$$

One can rule out “candidate” constraint-preserving boundary conditions (Calabrese, OS, *J. Math. Phys.* **44**, 3888 (2003)).
Such ill posed solutions can be constraint-violating or gauge modes!

Numerical results

- 3D numerical finite-difference code (Lehner, Nielsen, Tiglio)
- Domain is a cubic box $[-1,1]^3$.
- Third order Runge-Kutta time-discretization.
- Second-order accurate finite differencing for spatial operators.
- Some artificial dissipation.

Numerical results



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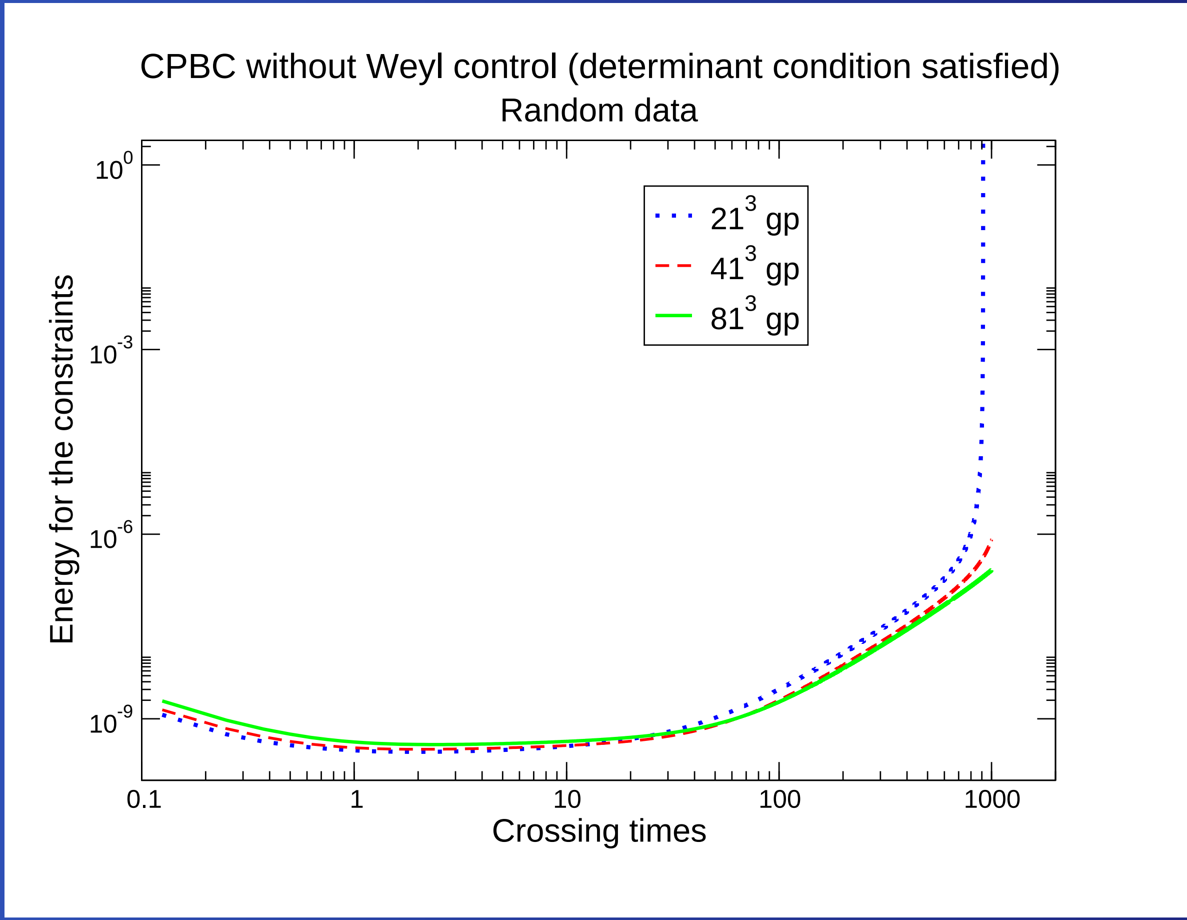

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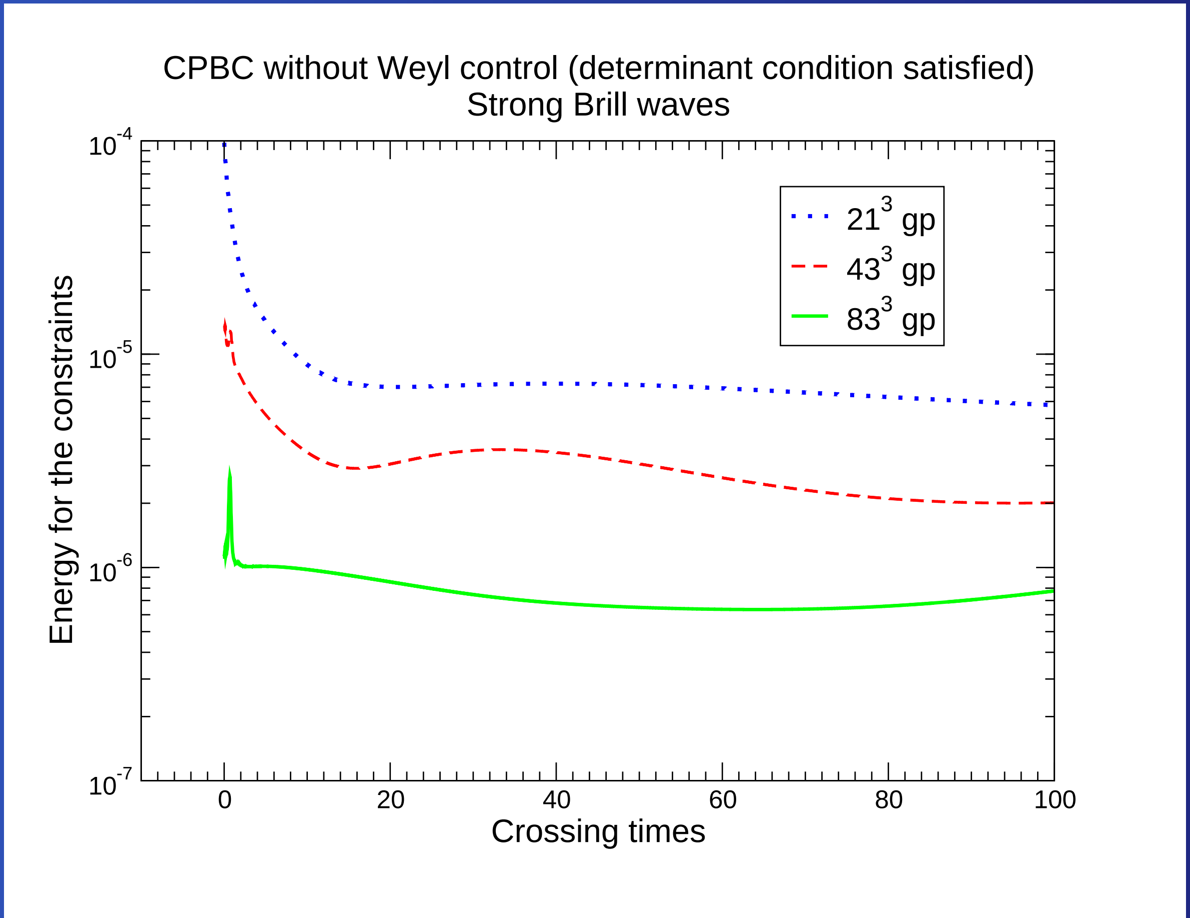
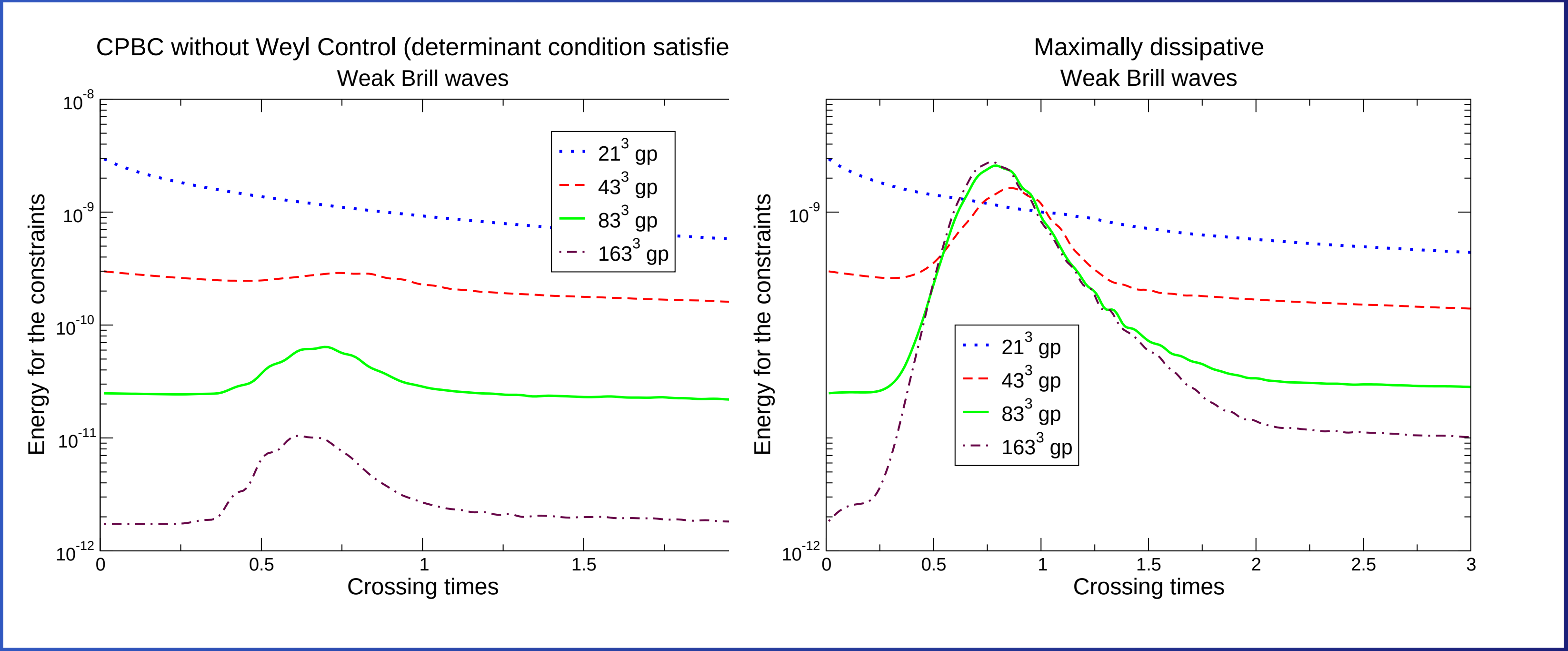


Numerical results

Comparison with non-constraint-preserving boundary conditions



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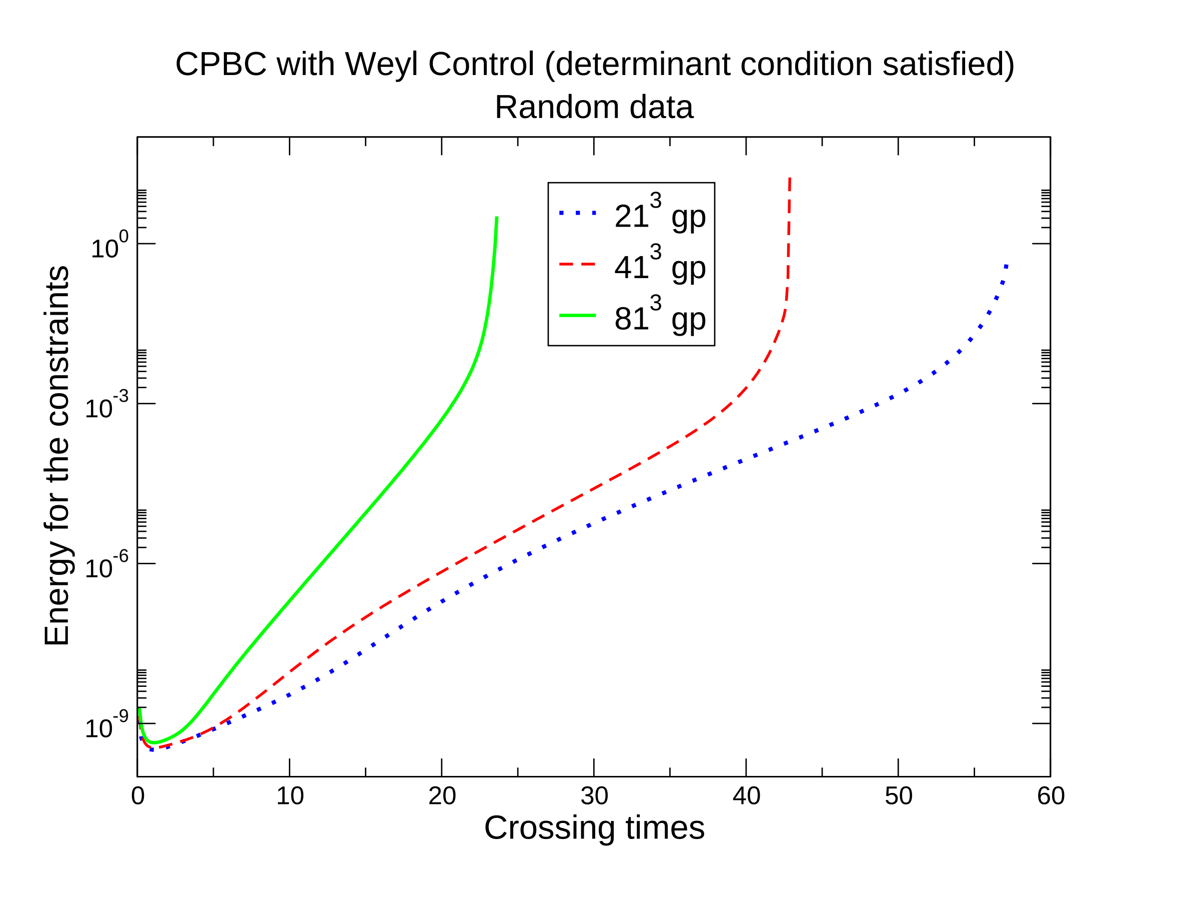

A related toy model in ED

Fat Maxwell ($A_i \leftrightarrow g_{ij}$, $E_j \leftrightarrow K_{ij}$, $W_{ij} \leftrightarrow d_{kij}$):

$$\begin{aligned} \partial_t A_i &= E_i + \nabla_i \phi, \\ \partial_t E_j &= \nabla^i (W_{ij} - W_{ji}) + \alpha \delta^{ij} C_{kij}, \\ \partial_t W_{ij} &= \nabla_i E_j + \frac{\beta}{2} \delta_{ij} \rho + \nabla_i \nabla_j \phi, \end{aligned}$$

with the constraints $\rho \equiv \nabla^k E_k = 0$, $C_{kij} = \nabla_k W_{ij} - \nabla_i W_{kj} = 0$.
Strongly hyperbolic if $\alpha\beta > 0$ (Cauchy problem well posed in L^2).
If boundaries are present, impose the boundary conditions

$\nabla^k E_k = 0$ preserves the constraints

$\mathbf{E}_{||} = (W_{n||} - W_{||n}) + h_{||}$ controls normal component of Poynting vecto

A related toy model in ED

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Choose the gauge condition $\phi = 0$ (temporal gauge $\leftrightarrow$ fixed shift).

- Well posed in L^2 ($u = (A_i, E_j, W_{ij})$)?

$$\|u(t,.)\|_{L^2(\Omega)} \leq a e^{bt} \left[\|u(0,.)\|_{L^2(\Omega)} + \int_0^t \|h(s)\|_{L^2(\partial\Omega)} \, ds \right].$$

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- The system passes the determinant condition for all $\alpha\beta > 0$
- However, consider solutions of the type

$$A_i = t\nabla_i f, \qquad E_j = \nabla_i f, \qquad W_{ij} = t\nabla_i \nabla_j f,$$

where f is a smooth, time-independent, harmonic function. Evolution and constraints equations are satisfied. Initial and boundary data only depend on first derivatives of f whereas the solution depends on second derivatives of f.

A related toy model in ED

- This is due to a bad gauge choice at the boundary! (physically one has an electrostatic solution with nontrivial electric charge density at the boundary)
- This motivates the following gauge condition:

$$\Delta\phi = -\nabla^k E_k, \qquad \text{on boundary: } \partial_n \phi = -E_n \ .$$

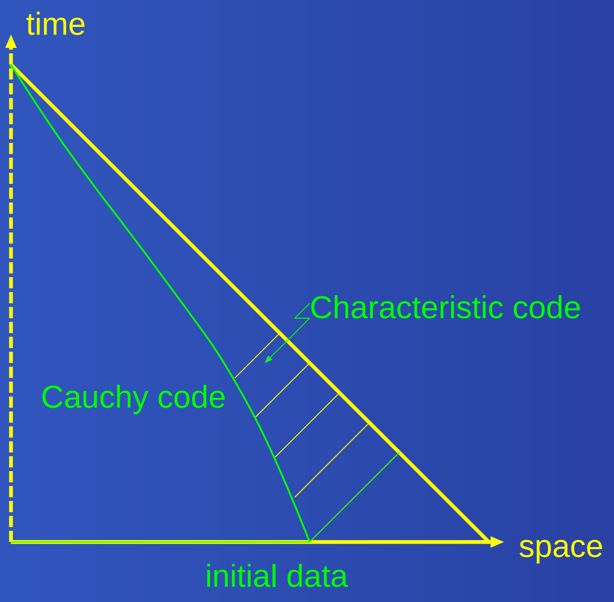
  Using this gauge condition, one can show that the problem is well posed in a Hilbert space that controls the L^2 norm of the fields *and* the constraint variables (Reula, S, gr-qc/0409027) Solution flux in this space is given by a semigroup.
- Current work with G. Nagy for generalization to Einstein (maximal slicing and minimal strain).

Conclusions

At the end of the day ???

Matching to a characteristic code

(Bishop, Winicour, d'Inverno, ...)



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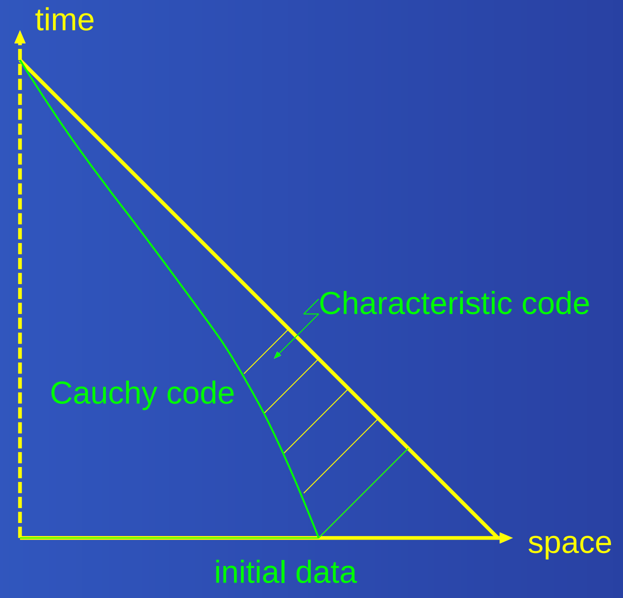

Conformal field equations

(Friedrich et al.)