What is Hyperbolicity and Why Should I Care?

No complete theory of hyperbolicity if 2nd derivatives

Can transform to 1st order using new variables

Hyperbolicity depends only on the *principal part*

$$\partial_t u + A^i \partial_i u = F$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}$$ vector of fundamental variables

A^i, F matrices

Characteristics

Characteristic matrix in direction n_i:

$$C \equiv A^i n_i$$

Characteristic speeds in direction n_i: eigenvalues of C

Can diagonalize C with matrix of eigenvectors: $S C S^{-1}$
Characterisic variables: linear combinations of fundamental variables $S u$

Propagate independently with characteristic speeds

Hyperbolicity

- Weakly hyperbolic: speeds all real for all directions n_i
- Strongly hyperbolic: + C has complete set of eigenvectors
  + (in $>$ 1-D) matrix of eigenvectors and its inverse bounded
- Symmetric hyperbolic: C symmetric for all directions n_i

Note: Strongly hyperbolic can be symmetrized, but symmetrizer depends on n_i

Well-Posedness

- Weakly hyperbolic typically ill-posed because of low order terms
- Strongly hyperbolic well-posed:

$$||u(t, x_i)|| \leq f(t)||u(0, x_i)||$$

($f(t)$ can blow up at finite t for nonlinear problems: shocks in fluids, singularities in GR)

Bounded solution exists for (small) finite time

For linear problems, $f(t) = Ke^{\alpha t}$

Not helpful numerically (use properties of actual problem?)

But ill-posed even worse!

Note: In general, need to consider the b.c.'s as well for well-posedness

Example — The Wave Equation

$$\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi$$

$$\partial_t \psi = -\Pi$$

$$\partial_i \psi = \Phi_i$$

(New variables $\rightarrow$ 1st order)

$$\partial_t \psi = -\Pi$$

$$\partial_t \Pi = \partial_i \Phi_i$$

$$\partial_t \Phi_i = \partial_i \Pi$$

Constraint: $\partial_i \psi = \Phi_i$

$$\partial_t \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \partial_y \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \partial_z \begin{pmatrix} \psi \\ \Pi \\ \Phi_x \\ \Phi_y \\ \Phi_z \end{pmatrix} = \begin{pmatrix} -\Pi \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C = A^i n_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_x & n_y & n_z \\ 0 & n_x & 0 & 0 & 0 \\ 0 & n_y & 0 & 0 & 0 \\ 0 & n_z & 0 & 0 & 0 \end{pmatrix}$$ (symmetric)

Eigenvalues: $\lambda = \pm 1$, 0 (3 times)

Eigenvectors for $\vec{n} = (1, 0, 0)$ ($\lambda = 0$):

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Characteristic variables: ψ, Φ_y, Φ_z

Eigenvectors for $\vec{n} = (1, 0, 0)$ ($\lambda = \pm 1$):

$$u = \begin{pmatrix} 0 \\ 1 \\ \mp 1 \\ 0 \\ 0 \end{pmatrix}$$

Characteristic variables: $\Pi \mp \Phi_x$

Check: if $\psi = \psi(t - x)$, no $\psi(t + x)$, then $\partial_t \psi - \partial_x \psi = 0$

i.e.,

$$\Pi + \Phi_x = 0$$

$$\Pi - \Phi_x \neq 0$$

Einstein's Equations

Traditional $3+1$: Write in 1st order form using

$$d_{kij} \equiv \partial_k g_{ij} \qquad (\rightarrow \text{ new constraint})$$

$$\partial_0 g_{ij} = \ldots$$
$$\partial_0 K_{ij} = \ldots$$
$$\partial_0 d_{kij} = \ldots$$

Find: characteristic speeds $\{0, \pm 1\}$
but no complete set of eigenvectors

Only weakly hyperbolic $\rightarrow$ ill-posed
Numerical experiments etc.: Calabrese et al. (2002)

Getting Strongly Hyperbolic Formulations

Require physical speeds $\{0, \pm 1\}$

Two ingredients:

1. Transform the lapse

2. Add multiples of constraints to evolution equations

Lapse: e.g., multiply by $g^{-\sigma}$ (other possibilities)

Find $g^{-1/2}$ necessary for physical speeds

Constraints: e.g. Maxwell

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} + \kappa \nabla \cdot \vec{B} + \zeta \nabla \cdot \vec{E}$$

Hamiltonian, momentum constraints

Additional constraints from new variables

Lots of possiblities!

Kidder, Scheel & Teukolsky (2001): σ + 4 parameters for constraints

Physical speeds $\rightarrow$ 2 free

Also, can redefine variables:

$$P_{ij} = K_{ij} + z g_{ij} K$$

(Play around with all traces)

$\rightarrow$ 7 more parameters

Do not affect speeds, but eigenvectors (stability, nonlinear terms)

Special cases: Fritelli-Reula
Einstein-Christoffel
...

Advantages of Hyperbolic Formulations of GR

- Simpler equations (wave equations!).

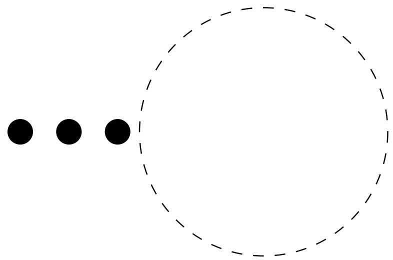
- Well-posed.

  (Note: does *not* exclude exponential growth.)

- Characteristic fields $\Rightarrow$
  Knowledge of allowed boundary conditions.

  Horizon excision well-defined.
  Well-defined prescription for artificial outer boundary.

  Spectral methods!



Disadvantages of Hyperbolic Formulations of GR

- More variables.
- More constraints must be satisfied.
- More complicated non-principal terms.

Example of hyperbolic formulation of GR

Einstein-Christoffel (Anderson & York 1999) (30 variables):

$$\hat{\partial}_0 g_{ij} = -2\alpha K_{ij}$$

$$\hat{\partial}_0 K_{ij} + \alpha\partial^k f_{kij} = \alpha X(g_{ij}, K_{ij}, f_{kij})$$

$$\hat{\partial}_0 f_{kij} + \alpha\partial_k K_{ij} = \alpha Y(g_{ij}, K_{ij}, f_{kij})$$

where

$$f_{kij} \equiv \Gamma_{(ij)k} + g^{\ell m}(g_{ki}\Gamma_{[\ell j]m} + g_{kj}\Gamma_{[\ell i]m})$$

ADM:

$$\hat{\partial}_0 g_{ij} = -2\alpha K_{ij}$$

$$\hat{\partial}_0 K_{ij} = \frac{\alpha}{2} g^{k\ell}\left(\partial_k\partial_\ell g_{ij} + \partial_i\partial_j g_{k\ell} - \partial_i\partial_\ell g_{kj} - \partial_k\partial_j g_{i\ell}\right) + \alpha F(g_{ij}, K_{ij}, \partial_k g_{ij})$$

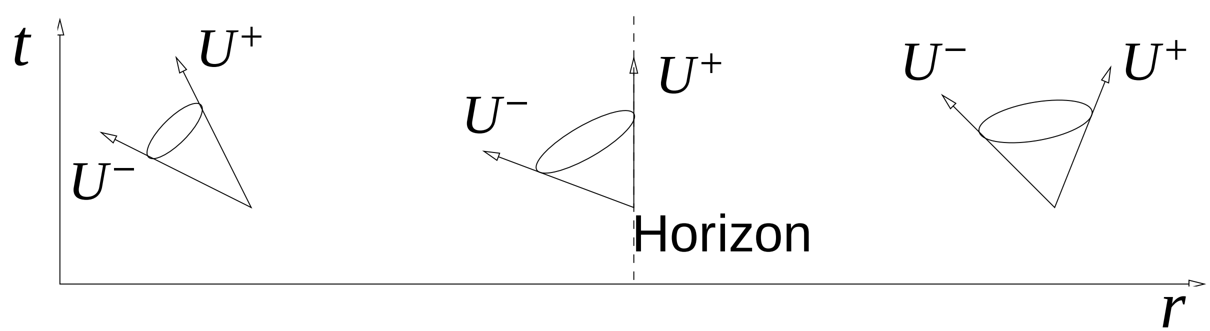
Spherical Symmetry (1D)

(Kidder, Scheel, Teukolsky, Carlson & Cook 2000)

- Einstein-Christoffel formulation.
  - 6 dynamical variables : $g_{rr}, g_{\theta\theta}, K_{rr}, K_{\theta\theta}, f_{rrr}, f_{r\theta\theta}$.
  - 6 characteristic fields:

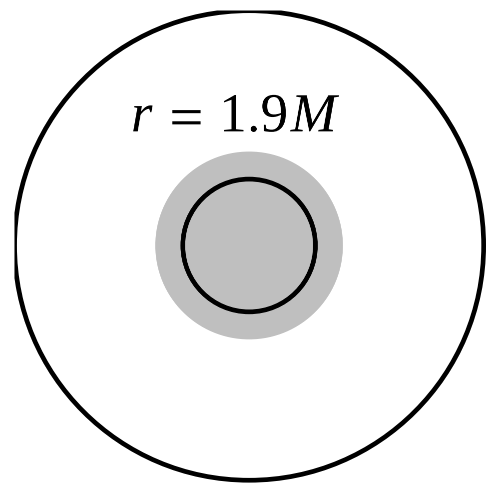
$$U_r^{\pm} \equiv K_{rr} \pm \frac{f_{rrr}}{\sqrt{g_{rr}}}, \qquad U_r^0 \equiv g_{rr},$$

$$U_\theta^{\pm} \equiv K_{\theta\theta} \pm \frac{f_{r\theta\theta}}{\sqrt{g_{rr}}}, \qquad U_\theta^0 \equiv g_{\theta\theta}.$$



- Use Chebyshev polynomials in r as basis.

Test Case: Schwarzschild BH, Spherical Symmetry

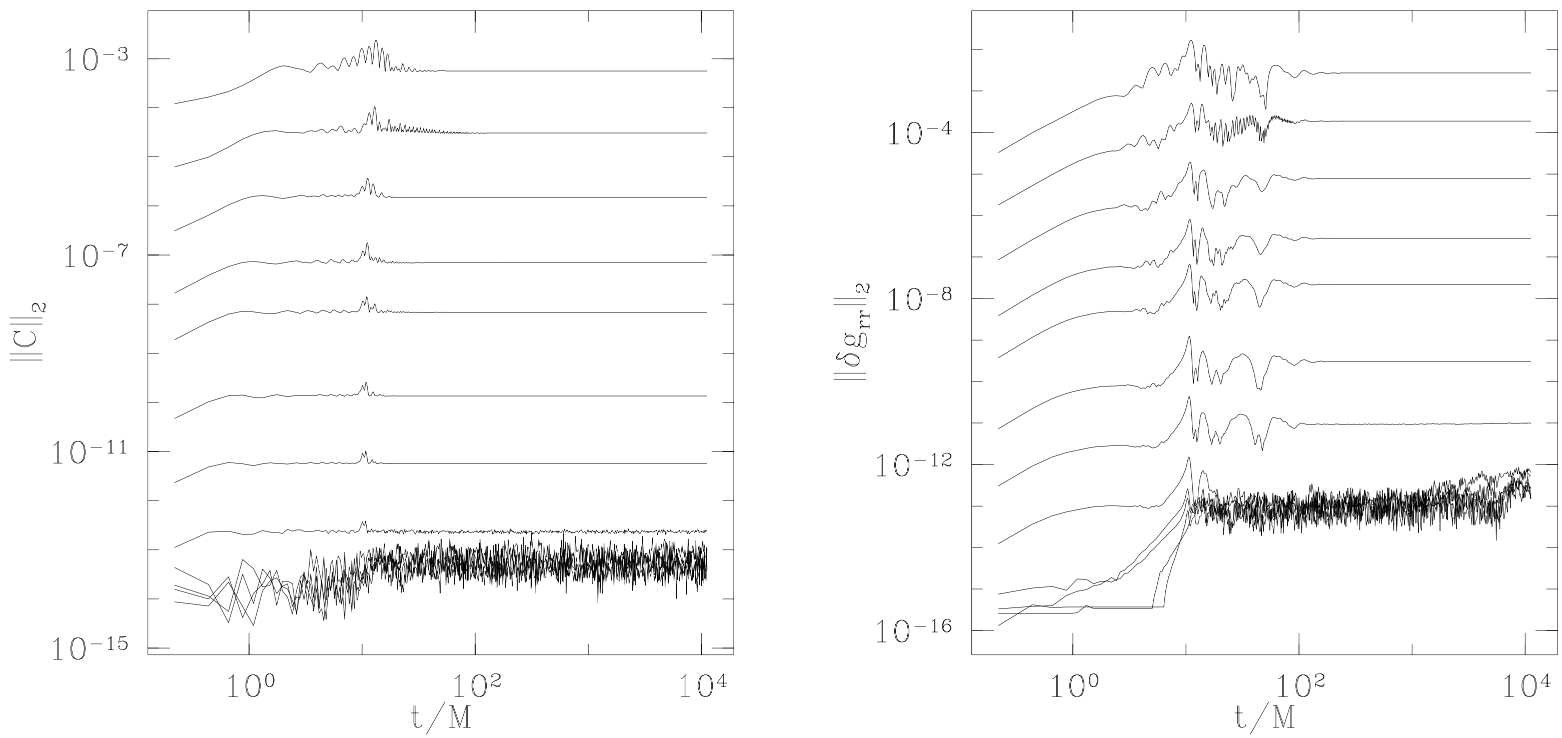


- Kerr-Schild coordinates
- Analytic gauge conditions.
- Inner boundary: No BC.
- Outer boundary:

  $U_r^0, U_\theta^0, U_r^-, U_\theta^-$ frozen.
  U_r^+, U_θ^+ untouched.

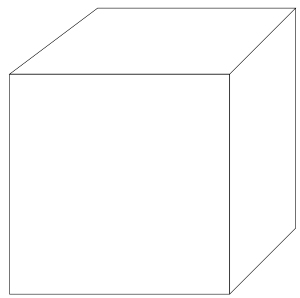
- 4th-order Runge-Kutta, $\Delta t = 0.007M$.

- Radial resolutions $12, 16, 20, 24, 27, 32, 36, 40, 45, 48, 54, 60.$

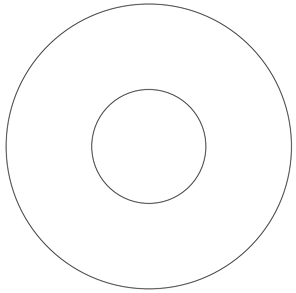


3D Black Hole Evolutions

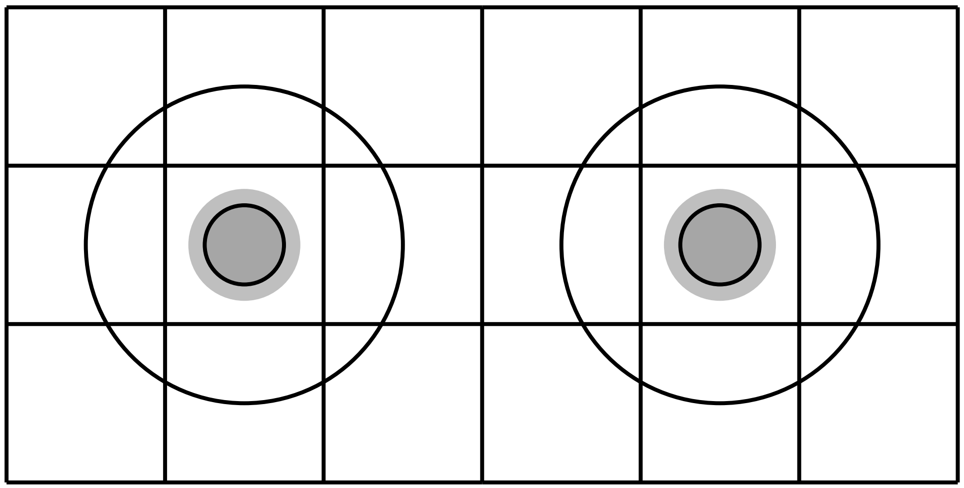
- Rectangular domains:
  - x,y,z basis: Chebyshev polynomials.



- Spherical shell domains:
  - r basis: Chebyshev polynomials.
  - Angular basis: $Y_{\ell m}(\theta, \phi)$.
  - Evolve *Cartesian* components.



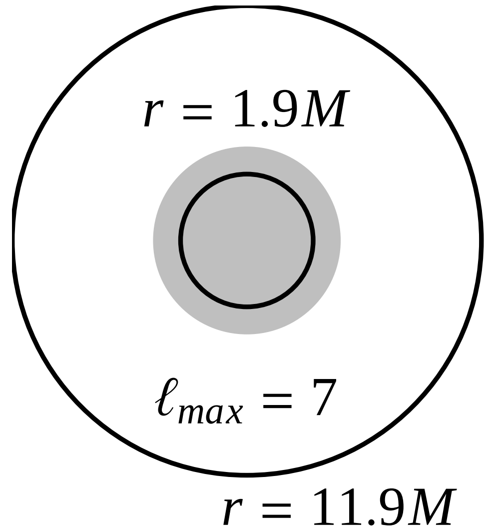
- How do we handle 2 black holes?
  Multiple domains:



- Interdomain boundaries:

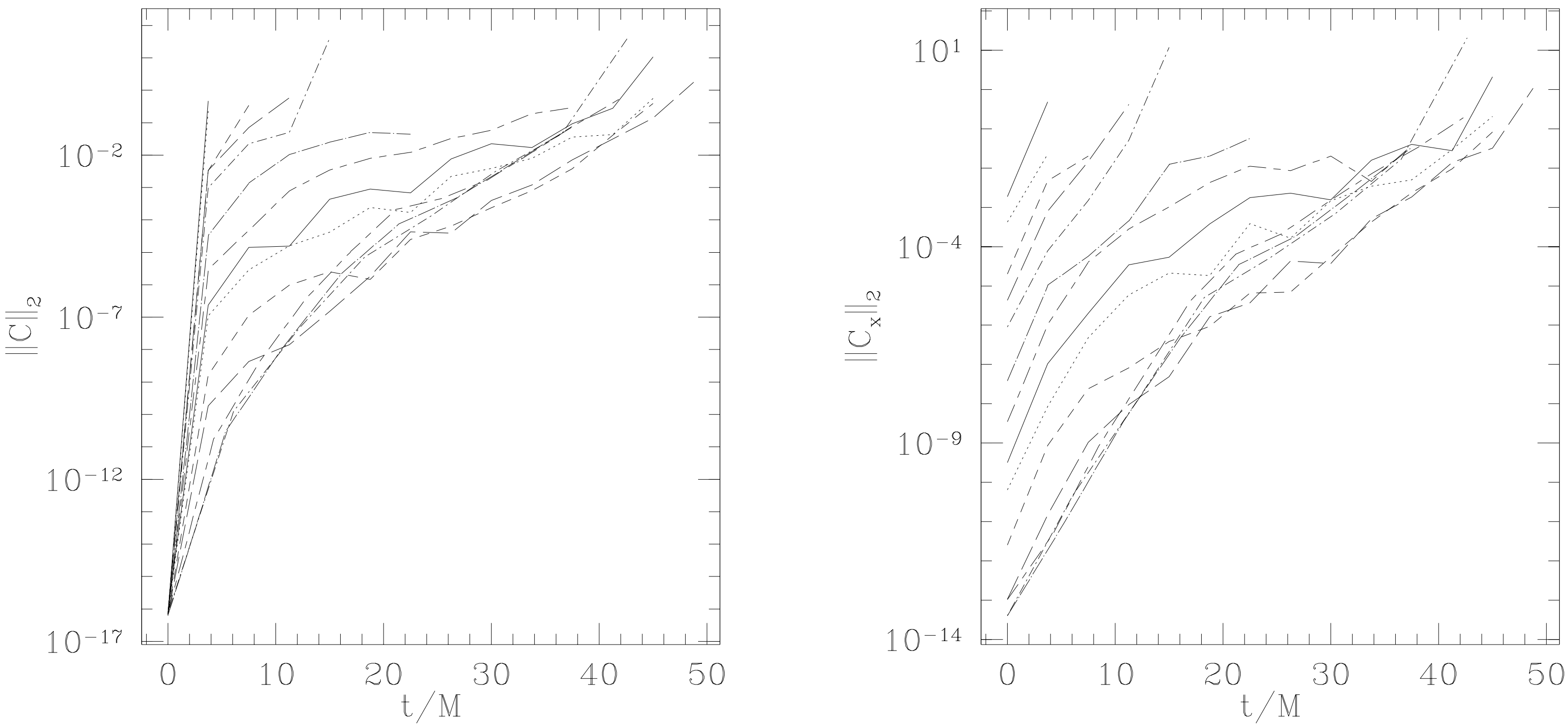
  Match characteristic fields (interpolate for overlapping domains).

Test Case: Single Schwarzschild BH, 3D



- Painlevé-Gullstrand coordinates.
- Analytic gauge conditions.
- Inner boundary: No BC
- Outer boundary:
  - Ingoing fields frozen.
  - Outgoing fields untouched.
- 4th-order Runge-Kutta, $\Delta t = 0.0075M$

- Radial resolutions
  8, 10, 12, 16, 18, 20, 24, 27, 30, 32, 36, 40, 45, 48, 54.



The Analytic Problem

Einstein’s equations are an overdetermined system

Pick a subset to solve (e.g., evolution equations)

- “All” 3-d black hole evolutions unstable
  (Important accomplishment of past decade)

How to deal with this . . .

- New formulations
- Control the instability
  - λ system (Brodbeck et al. 1999)
  - Dynamical feedback of parameters (Tiglio 2003)
- Reimpose constraints (fast elliptic solver)
- . . .

Extended Einstein-Christoffel system

Starting with ADM system:

- Densitize lapse

$$Q \equiv \log\left(\alpha g^{-1/2}\right)$$

- Define new variables

$$P_{ij} \equiv K_{ij} + z g_{ij} K$$

$$M_{kij} \equiv \frac{1}{2}\left\{\partial_k g_{ij} + g_{ij} g^{mn}\left[a \partial_k g_{mn} \quad + b \partial_m g_{kn}\right] + g_{k(i} g^{mn}\left[c\, \partial_{j)} g_{mn} + d \partial_m g_{j)n}\right]\right\}$$

- Write down evolution equations, add constraints (note: $\hat{\partial}_0 \equiv \partial_t - £_\beta$)

$$\hat{\partial}_0 P_{ij} = (\ldots) + \gamma N g_{ij} C$$

$$\hat{\partial}_0 M_{kij} = (\ldots) + \eta N g_{k(i} C_{j)} + \chi N g_{ij} C_k$$

- Demand simple wave equations of form

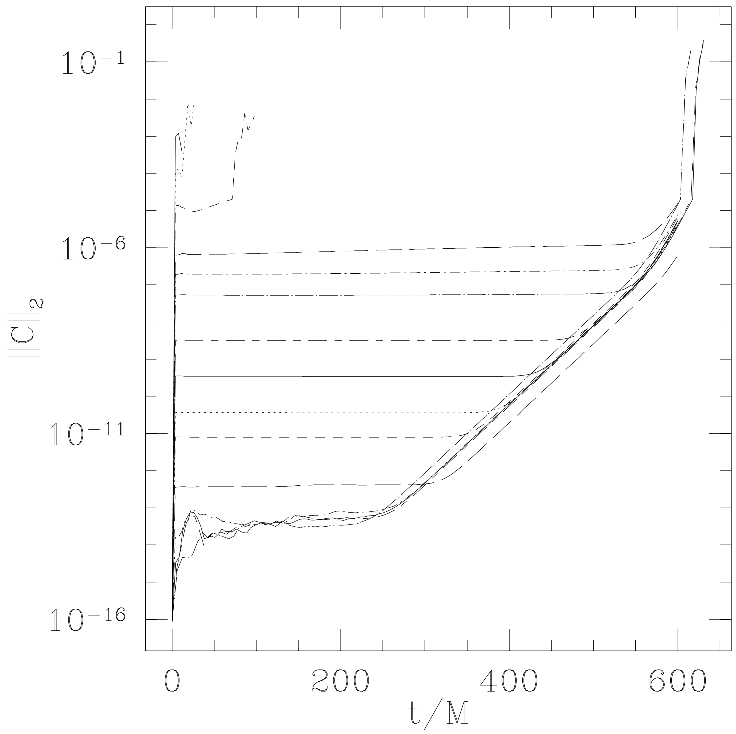
$$\hat{\partial}_0 P_{ij} + \alpha \partial^k M_{kij} = \alpha X(g_{ij}, P_{ij}, M_{kij}, \partial_i Q, \partial_i \partial_j Q)$$
$$\hat{\partial}_0 M_{kij} + \alpha \partial_k P_{ij} = \alpha Y(g_{ij}, P_{ij}, M_{kij}, \partial_i Q)$$

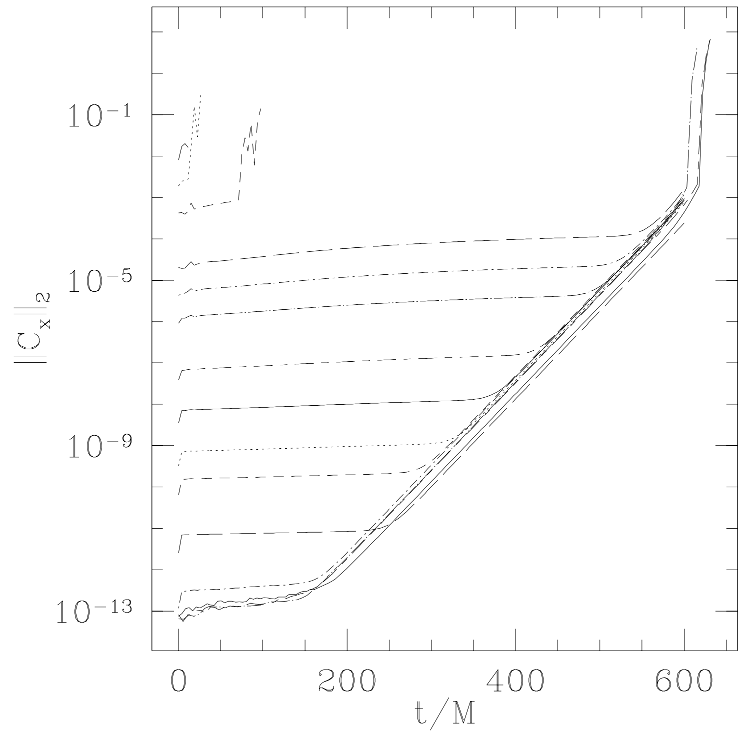
$\Rightarrow$ Only 2 undetermined parameters: z,γ.

- For $z = \gamma = 0$, recover Einstein-Christoffel

Extended EC Results, 3D

$$z = -1/4, \gamma = -2$$





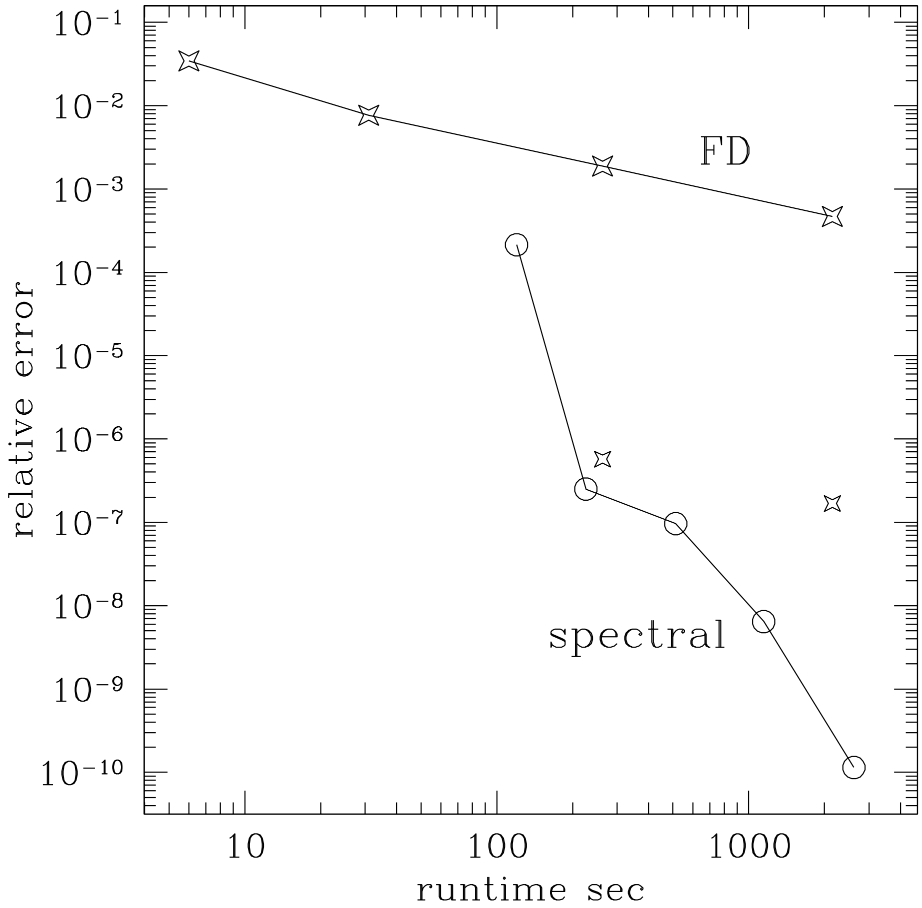
Solving the Initial Value Equations

Elliptic equations traditionally expensive to solve

New elliptic solver (Pfeiffer et al. 2003):

- Multidomain spectral code
- Very fast! (Solve constraints during evolution?)
- Can handle varying scales, e.g. $M_1/M_2 \sim 100$.

Speed Comparison



$$\nabla^2\psi + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

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