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$$\psi(R) g_{\alpha\beta} = g_{\alpha\beta}, \qquad (2.1)$$

One can pick local charts for which the metric has the form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2; \qquad (2.2)$$

or (with $r \to \exp \int^r r^{-1} e^{\lambda} dr$), one obtains the isotropic form

$$ds^2 = -\alpha^2 dt^2 + \psi^4 (dr^2 + r^2 d\Omega^2). \qquad (2.3)$$

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Static, spherically symmetric spacetimes

As you know, the Einstein tensor $G_{\alpha\beta}$ has as its only nonvanishing components

$$G^t{}_t = e^{-2\lambda}(-\frac{2}{r}\lambda' + \frac{1}{r^2}) - \frac{1}{r^2} = -\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\lambda})]$$

$$G^r{}_r = e^{-2\lambda}(\frac{2}{r}\Phi' + \frac{1}{r^2}) - \frac{1}{r^2} \qquad (2.4)$$

$$G^\theta{}_\theta = G^\phi{}_\phi = -e^{-2\lambda}[\Phi' + (\phi')^2 + \frac{1}{r}(\Phi' - \lambda') - \Phi'\lambda'], \qquad (2.5)$$

with vacuum solution,

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2, \qquad r \neq 2M. \tag{2.6}$$

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From $u^\alpha = u^t t^\alpha$ and $u^\alpha u_\alpha = -1$ we have

$$u^\alpha = e^{-\Phi} t^\alpha. \tag{2.7}$$

Hydrostatic equilibrium is then

$$\log \mathcal{E} = \log h - \log u^t = \int_0^P \frac{dP}{\varepsilon + P} + \Phi, \tag{2.8}$$

or

$$\Phi' = -\frac{P'}{\varepsilon + P}\,. \tag{2.9}$$

The $G_t^t = 8\pi T_t^t$ and $G^r{}_r = 8\pi T^r{}_r$ equation give λ and Φ in the form,

$$e^{2\lambda} = \frac{1}{1-\frac{2m}{r}}, \text{ with } m = \int_0^r \varepsilon\, 4\pi r^2 dr. \tag{2.10}$$

$$\Phi' = \frac{m + 4\pi P r^3}{r(r - 2m)}. \tag{2.11}$$

Equating the two expressions for Φ' in (2.11) and (2.9), we obtain the equation of hydrostatic equilibrium, the Tolman-Oppenheimer-Volkov Equation,

$$\frac{dP}{dr} = -(\varepsilon + P)\frac{m + 4\pi P r^3}{r(r - 2m)} \quad . \tag{2.12}$$

Note that the Newtonian limit $(P \ll \varepsilon, R \ll M)$ of (2.11) is

$$\Phi' = \frac{m}{r^2} \ ,$$

so that Φ becomes the Newtonian potential.

One obtains a barotropic star by integrating Eq. (2.12) and the defining equation (2.10) for m, with a given equation of state $P = P(\varepsilon)$. Explicitly, one begins with a central density ε_c and integrates up to the radius R at which P drops to zero (P is a decreasing function of r).

Φ is fixed outside the star by $\Phi = -\lambda$, inside by $\mathcal{E} = \Phi_{\mathcal{S}}$,
$\Phi = \mathcal{E} - \log h$.

Rotating Relativistic Stars

The metric $g_{\alpha\beta}$ of a stationary axisymmetric rotating fluid has two commuting Killing vectors, ϕ^α and t^α , generating rotations and time-translations. (t^α agrees asymptotically with time-translation, but within an ergosphere or horizon, t^α will be spacelike.) As before, the fluid velocity has the form

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha),$$

and the equation of hydrostatic equilibrium has the first integral

$$\frac{h}{u^t} = \mathcal{E} = \text{ constant.}$$

Geometry of a Rotating Star

The metric, $g_{\alpha\beta}$, can be written in terms of dot products of the Killing vectors,

$$t^\alpha t_\alpha, \qquad t^\alpha \phi_\alpha, \qquad \phi^\alpha \phi_\alpha, \tag{2.13}$$

and a conformal factor, $e^{2\mu}$, that characterizes the geometry of the orthogonal 2-surfaces:

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and a conformal factor, $e^{2\mu}$, that characterizes the geometry of the orthogonal 2-surfaces:

$$g^{tt} = \nabla_\alpha t \nabla^\alpha t = -e^{-2\nu},$$

$$
\begin{aligned}
g_{\phi\phi} &= \phi^{\alpha}\phi_{\alpha} = e^{2\psi}, \\
g_{t\phi} &= t^{\alpha}\phi_{\alpha} = -\omega e^{2\psi}.
\end{aligned} \tag{2.14}
$$

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\end{aligned} \tag{2.14}
$$

Then

$$
g_{tt} = t^\alpha t_\alpha = -e^{2\nu} + \omega^2 e^{2\psi}, \tag{2.15}
$$

and

$$
ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2), \tag{2.16}
$$

where ϖ and z are cylindrical coordinates labeling the 2-surfaces orthogonal to t^α and ϕ^α.

The Killing vectors have components,

$$
t^\mu = \delta^\mu_t, \qquad \phi^\mu = \delta^\mu_\phi \tag{2.17}
$$

and the symmetry means that the potentials ν, ψ, ω and μ depend only on ϖ and z. Because of the choice of an overall conformal factor, $e^{2\mu}$, to describe the geometry of the $\varpi - z$ surfaces, the exterior of a spherical star given by Eq. (2.16) is the

Schwarzschild geometry in isotropic coordinates,

$$e^{\nu} = \frac{1 - M/2r}{1 + M/2r}, \quad e^{\psi} = \varpi(1 + M/2r)^2, \quad e^{\mu} = (1 + M/2r)^2. \tag{2.18}$$

Asymptotically, the relations

$$e^{\psi} = \varpi(e^{-\nu} + O(r^{-2})), \qquad e^{\mu} = e^{-\nu} + O(r^{-2}), \tag{2.19}$$

hold for the potentials, (2.18), and for the metric (2.16) as well, because any stationary, asymptotically flat spacetime agrees with the Schwarzschild geometry to order r^{-1}. If, following Bardeen and Wagoner (1971), we write

$$\beta := \psi + \nu, \qquad \zeta := \mu + \nu, \tag{2.20}$$

then, asymptotically, ζ, which vanishes for isotropic Schwarzschild, is itself of order r^{-2}.

The angular velocity $\omega \equiv -t^{\alpha}\phi_{\alpha}/\phi^{\beta}\phi_{\beta}$, measures the dragging of inertial frames in the sense that particles with zero angular momentum move along trajectories whose angular velocity relative to infinity is $d\phi/dt = \omega$. A natural tetrad is the frame of zero-angular-momentum-observers (ZAMOs), with basis covectors

$$\omega^{(0)} = e^{\nu} dt, \quad \omega^{(1)} = e^{\psi}(d\phi - \omega dt), \quad \omega^{(2)} = e^{\mu} d\varpi, \quad \omega^{(3)} = e^{\mu} dz, \tag{2.21}$$

and the corresponding contravariant basis vectors are

$$e_{(0)} = e^{-\nu}(\partial_t + \omega\partial_\phi), \quad e_{(1)} = e^{-\psi}\partial_\phi, \quad e_{(2)} = e^{-\mu}\partial_\varpi, \quad e_{(3)} = e^{-\mu}\partial_z. \tag{2.22}$$

The nonzero components of the four velocity u^α along these frame vectors can be written in terms of a fluid 3-velocity v in the manner

$$u^{(0)} = \frac{1}{\sqrt{1-v^2}}, \qquad u^{(1)} = \frac{1}{\sqrt{1-v^2}}. \tag{2.23}$$

Then

$$u^t = u^\alpha \nabla_\alpha t = \frac{e^{-\nu}}{\sqrt{1-v^2}}, \qquad u^\phi = u^\alpha \nabla_\alpha \phi = \Omega u^t, \tag{2.24}$$

where Ω is the angular velocity of the fluid relative to infinity (measured by an asymptotic observer with 4-velocity along the asymptotically timelike Killing vector, t^α). The 3-velocity, v, written in terms of Ω, is

$$v = e^{\psi-\nu}(\Omega - \omega). \tag{2.25}$$

Note that $2\pi e^\psi$ is the circumference of a circle centered about the axis of symmetry (the z-axis); that is, e^ψ agrees for spherical stars with $r \sin\theta$, where r and θ are the usual Schwarzschild coordinates (not the isotropic coordinates introduced above).

The nonvanishing tetrad components of $T^{\alpha\beta}$ are

$$T^{(0)(0)} = \frac{\varepsilon + p v^2}{1 - v^2}, \qquad T^{(0)(1)} = \varepsilon + p \frac{v}{1 - v^2}, \tag{2.26}$$

$$T^{(1)(1)} = \frac{\varepsilon v^2 + p}{1 - v^2}, \qquad T^{(2)(2)} = T^{(3)(3)} = p. \tag{2.27}$$

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The four potentials are determined by four components of the field equation

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.28}$$

whose selection is a matter of taste. Following Bardeen and Wagoner (1971), Butterworth and Ipser (1976) and several subsequent authors based their code on the following four equations, in which ∇ is the *flat* 3-dimensional covariant derivative operator of the metric: $d\varpi^2 + dz^2 + \varpi^2 d\phi^2$.

$$e^{-\beta}\ R^{(0)(0)} = e^{-\beta+2\nu}R^{tt}:$$

$$\nabla^2\nu - \frac{1}{4}e^{\beta-4\nu}\nabla^a\omega\nabla_a\omega = -8\pi e^{-\beta}\Big[(\varepsilon+p)\frac{1+v^2}{1-v^2}+2p\Big]; \tag{2.29}$$

$$e^{\beta-2\nu}R^{(0)(1)} = 2R^t_\phi\ :$$

$$\nabla^a(e^{2\beta-4\nu}\nabla_a\omega) = -16\pi\, e^{\beta-2\nu}(\varepsilon+p)\frac{v}{1-v^2}; \tag{2.31}$$

$$e^{-\beta}(G^{(2)(2)}+G^{(3)(3)}) = e^{\beta-2\mu}(G_{\varpi\varpi}+G_{zz})$$

$$\nabla^a\nabla_a\beta = 16\pi e^{-\beta}p; \tag{2.32}$$

and

$$e^{2\mu}G^{(2)(3)} = G_{\varpi z}:$$

$$\mu_{,\varpi}\beta_{,z}+\mu_{,z}\beta_{,\varpi} = \beta_{,\varpi z}+\beta_{,\varpi}\beta_{,z}+2\nu_{,\varpi}\nu_{,z}-\beta_{,z}\nu_{,\varpi}+\frac{1}{2}e^{2\beta-4\nu-2\mu}\omega_{,\varpi}\omega_{,z}.$$

Alternatively, one can use a 4th elliptic equation for μ.

Living Reviews: Nick Stergioulas

Codes by Wilson; Bonazzola & Schneider; Butterworth & Ipser; JF, Ipser, Parker; Lattimer et al; Komatsu, Eriguchi, Hachisu; Cook, Shapiro, Teukolsky; Stergioulas (*rns, a public domain code, available at http://www.gravity.phys.uwm.edu/rns*); Bonazzola, Gourgoulhon, Salgado, Marck; Ansorg, Kleinwachter, Meinel.

	AKM	Lorene/ rotstar	SF (260x400)	SF (70x200)	BGSM	KEH
$\bar{p}_c$	1					
r_p/r_e	0.7				1e-3	
$\bar{\Omega}$	1.41170848318	9e-6	3e-4	3e-3	1e-2	1e-2
$\bar{M}$	0.135798178809	2e-4	2e-5	2e-3	9e-3	2e-2
$\bar{M}_0$	0.186338658186	2e-4	2e-4	3e-3	1e-2	2e-3
$\bar{R}_{circ}$	0.345476187602	5e-5	3e-5	5e-4	3e-3	1e-3
$\bar{J}$	0.0140585992949	2e-5	4e-4	5e-4	2e-2	2e-2
Z_p	1.70735395213	1e-5	4e-5	1e-4	2e-2	6e-2
Z_{eq}^f	-0.162534082217	2e-4	2e-3	2e-2	4e-2	2e-2
Z_{eq}^b	11.3539142587	7e-6	7e-5	1e-3	8e-2	2e-1
$\|\mathrm{GRV3}\|$	$4e-13$	3e-6	3e-5	1e-3	4e-3	1e-1

Code Comparison

Method:

1. Start with a guessed solution (e.g., for a spherical configuration).
   Solve the 4 field equations by Newton-Raphson, putting the linearized operator on the left side and the nonlinear terms on the right. (KEH solve by keeping only a flat-space laplacian on the each left side and solving by using the known Green's function).

2. Update h from the first integral of the equation of hydrostatic equilibrium, and use the EOS to find P, ε.

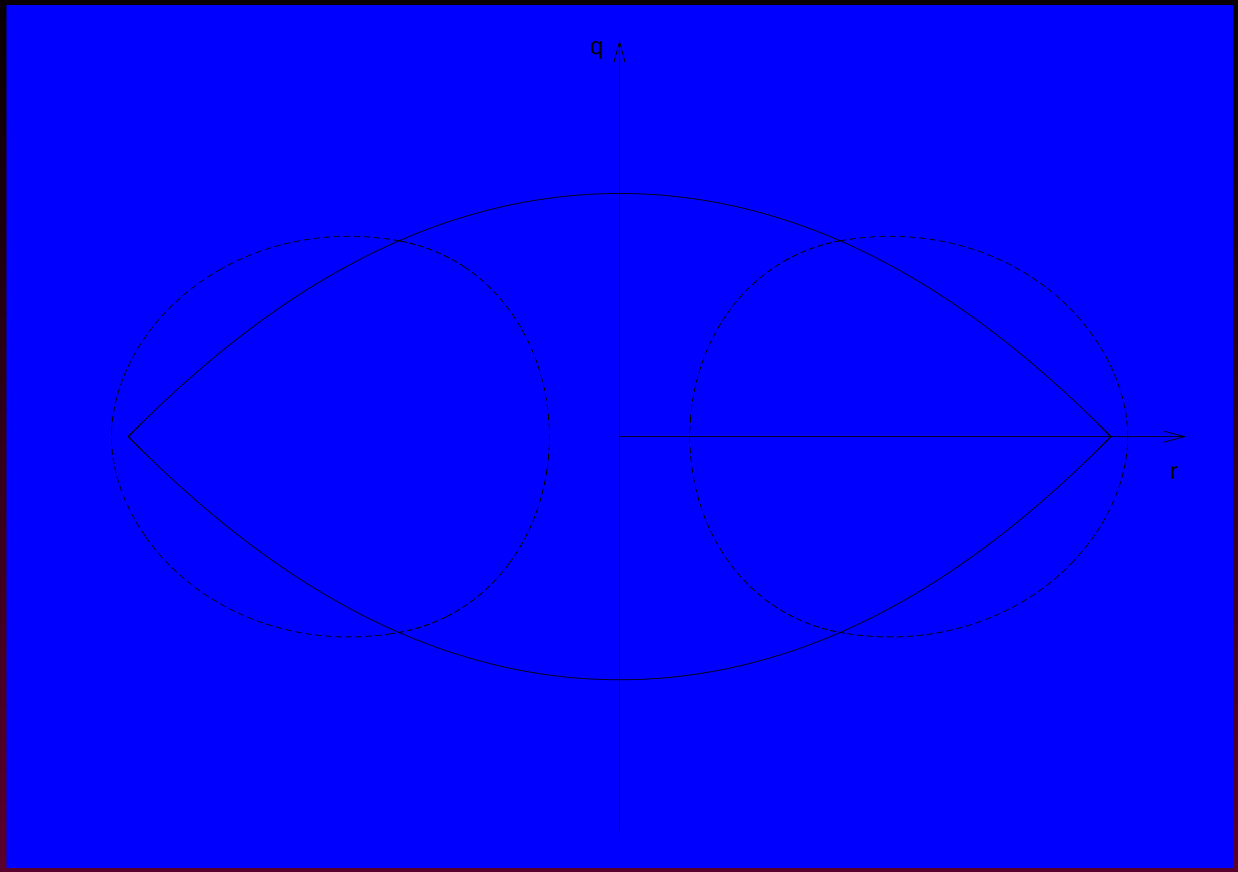
3. Find the new surface of the star.

4. Use the updated ε, P and the updated potentials to recompute the right-hand sides of the field equations.

$5 \equiv 1.$

Use spherical harmonics (Legendre polynomials) or Chebyshev polynomials for the θ dependence. For r dependence, directly specify function on the grid, using finite differences for radial derivatives, or use spectral decomposition with Chebyshev polynomials.

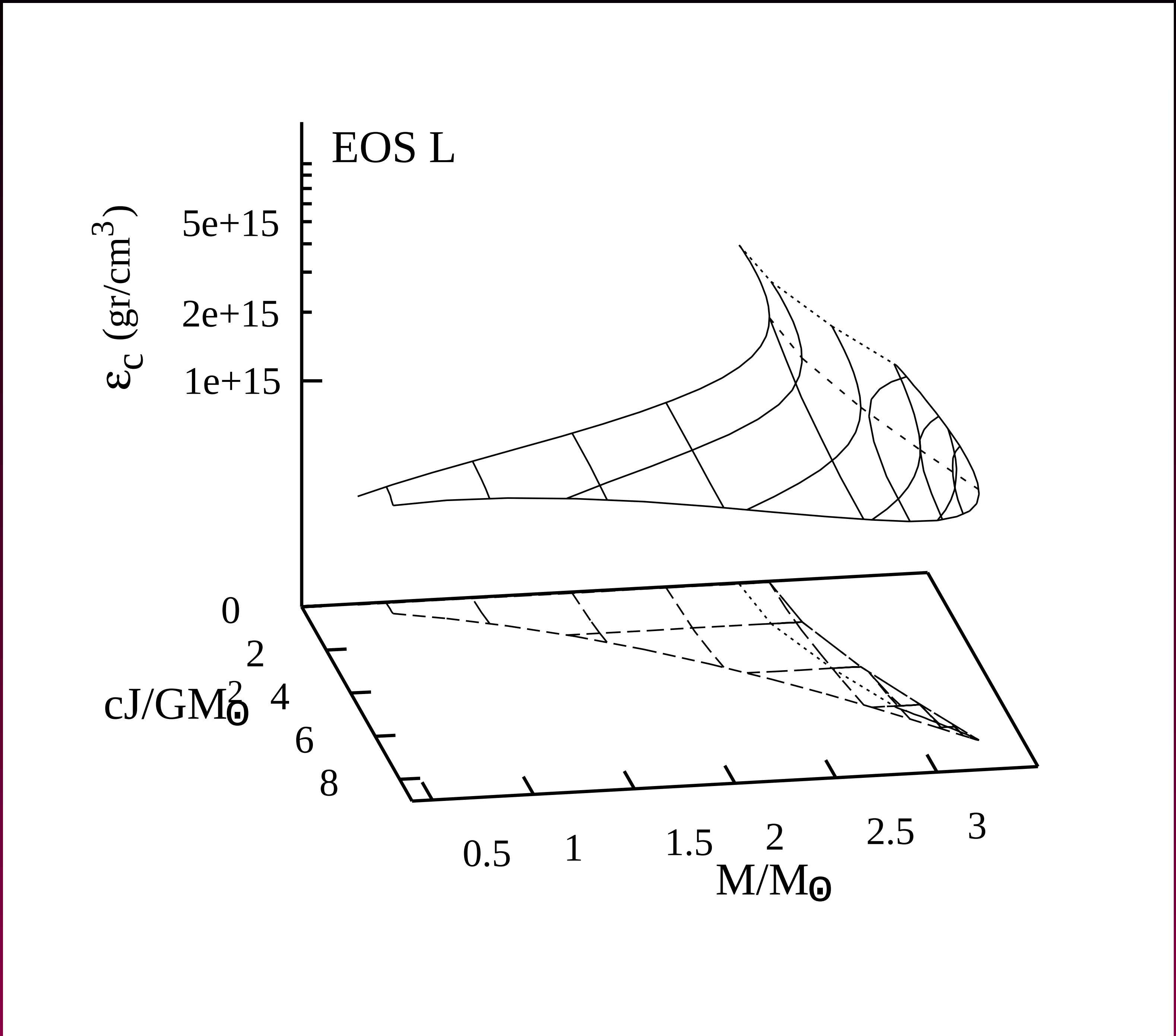
The accuracy of spectral methods was initially limited by the Gibbs phenomenon at the stellar surface, but the most recent spectral codes by the Meudon group and by Ansorg et al. overcome the problem by using two or three domains fitted to the stellar surface. Ansorg et al. obtain near-machine accuracy with two domains and a Chebyshev expansion for both r and θ.



Ansorg et al.'s model of a uniformly rotating, uniform-density star rotating at maximum angular velocity Ω_K: the star rotates at the angular velocity of a satellite in Keplerian orbit at the equator. The two lobes mark the boundaries of the ergosphere. Uniformly rotating stars with realistic equations of state reach Ω_K before an ergosphere appears.

The set of equilibrium configurations of a uniformly rotating star is two-dimensional, specified, for example, by M_0 and Ω. The 2-dimensional surface of equilibria shown on the next page is ruled by lines of constant J and M_0. For fixed J, the maximum mass configuration marks the onset of instability to collapse. This instability line is also the set of points points at which J is a maximum along a sequence of constant M_0.

(JF, Ipser, Sorkin; Cook, Shapiro, Teukolsky).



EOS L
ε_c (gr/cm^3)
5e+15
2e+15
1e+15
0
2
4
6
8
cJ/GM$_\odot^2$
0.5
1
1.5
2
2.5
3
M/M$_\odot$

Although not shown in the figure, at low density there is a similar line of minimum mass configurations. Below the minimum mass, configurations are unstable to explosion - they are unbound. Candidates for realistic equations of state typically have maximum masses for uniform rotation below $2.5M_\odot$.
A hard upper limit on the mass of uniformly rotating, self-gravitating stars is found by using the stiffest EOS consistent with causality ($v_{sound} = dP/d\varepsilon = 1$), matching at a density ε_m to a known low-density EOS.

$$M < 6.1M_\odot \left(\frac{2 \times 10^{14} \mathrm{g/cm}^3}{\varepsilon_m} \right)^{1/2} \tag{2.33}$$

JF,Ipser; Koranda, Stergioulas, JF (See, e.g. Cook, Shapiro, Teukolsky, for upper mass limits for a representative sample of candidate EOSs).

The next two sections were not part of the oral lectures and can be regarded as an appendix to topic II on stellar equilibria.

Injection Energy

The quantity $\mathcal{E}$ has a natural physical interpretation, as the energy per unit mass needed to inject matter into the star, with the injected fluid in the same local state (same composition, density, and entropy density as the surrounding star). We will compute the initial energy δM needed to inject a ring of fluid into a rotating star, after dropping it to its new location, a circle C about the axis of symmetry.

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Drop a box from infinity with energy δM, rest mass $\delta M_0 = m_B \delta N$, entropy δS , and angular momentum δJ.

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If the box has, at infinity, four-momentum p_α, then

$$\delta M = -p_\alpha t^\alpha, \qquad \delta J = p_\alpha \phi^\alpha.$$

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If the box has, at infinity, four-momentum p_α, then

$$\delta M = -p_\alpha t^\alpha, \qquad \delta J = p_\alpha \phi^\alpha.$$

When the freely falling box reaches a point in the star, its energy measured by an observer at rest with respect to the fluid is

$$\delta E = -p_\alpha u^\alpha,$$

where $u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)$ is the fluid four-velocity.

$$\begin{aligned} \delta E &= u^t(p_\alpha t^\alpha + \Omega p_\alpha \phi^\alpha) \\ &= u^t(\delta M - \Omega\delta J). \end{aligned} \tag{2.34}$$

The relation is more often written in the form

$$\delta M = \frac{\delta E}{u^t} + \Omega\delta J. \tag{2.35}$$

In adding baryon mass δN baryons, with locally measured energy $\frac{\varepsilon}{n}$ per baryon, one is directly adding an energy $\frac{\varepsilon}{n}\delta N$. From the 1st law of thermodynamics, if one adds δN baryons with entropy δS, one is adding energy δE given by

$$\delta E = T\delta S + \frac{\varepsilon + P}{n}\delta N.$$

or

$$\delta E = T\delta S + \frac{\varepsilon + P}{\rho}\delta M_0. \qquad (2.36)$$

Again the presence of $\varepsilon + P$ instead of ε arises from the work $P\delta V$ done to create a space δV for the new baryons. With $\delta M_0 = \rho\delta V$, we have $P\delta V = \frac{P}{\rho}\delta M_0$.

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Finally from Eqs. (2.35) and (2.37)

$$\delta M = \frac{\varepsilon + P}{\rho u^t}\delta M_0 + \frac{T}{u^t}\delta S + \Omega\delta J. \qquad (2.37)$$

The coefficient of δM_0 is the energy $\mathcal{E}$, the injection energy per unit rest mass of matter with zero initial entropy and angular momentum.

Why is $\mathcal{E}$ is constant in a star with constant entropy per baryon and constant angular velocity Ω? An equilibrium configuration is an extremum of mass at fixed angular momentum, entropy and baryon number: Small changes in the structure of the star leave the mass fixed. In particular, suppose one moves a ring of fluid from one location to another in a uniformly rotating white dwarf or neutron star, stars that are approximately barotropic because T is approximately zero (that is, $kT << \varepsilon_F$).

Changing the location of the ring is equivalent to moving it out to infinity and back in to a new location in the star. According to Eq. (2.37) with $T = 0$

$$\delta M = \left[\left(\frac{\varepsilon + P}{n u^t} \right)_2 - \left(\frac{\varepsilon + P}{n u^t} \right)_1 \right] \delta N + (\Omega_2 - \Omega_1) \delta J. \tag{2.38}$$

For uniformly rotating star, $\delta M = 0$ implies

$$\left(\frac{\varepsilon + P}{n u^t} \right)_2 = \left(\frac{\varepsilon + P}{n u^t} \right)_1 , \tag{2.39}$$

and we conclude that $\mathcal{E}$ is constant throughout the star.

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Changing the location of the ring is equivalent to moving it out to infinity and back in to a new location in the star. According to Eq. (2.37) with $T = 0$

$$\delta M = \left[\left(\frac{\varepsilon + P}{nu^t}\right)_2 - \left(\frac{\varepsilon + P}{nu^t}\right)_1\right]\delta N + (\Omega_2 - \Omega_1)\delta J. \qquad (2.38)$$

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and we conclude that $\mathcal{E}$ is constant throughout the star.

Variational Principle for Relativistic Fluids

We will show that a perfect fluid with EOS $\epsilon = \epsilon(P)$, has an action of the form

$$I_{\text{fluid}} = \int \epsilon\sqrt{-g}\,dx. \tag{2.40}$$

The action is a functional of the fluid's history. If one thinks of a fluid as a coherently moving collection of particles, a fluid configuration is specified by giving the location of each particle. Beginning with some (arbitrarily chosen) initial fluid spacetime, ϵ, P, u^α and metric $g_{\alpha\beta}$, one can specify another fluid spacetime with the same number of baryons by giving its metric $\bar{g}$ and a diffeo X that maps the original fluid to its new position.

A precise way of saying that an action I is an extremum is to say that for any smooth family of histories $X_\lambda, g_{\lambda\ \alpha\beta}$,

$$\delta I \equiv \frac{d}{d\lambda} I(\lambda) = 0,$$

with $I(\lambda) = I(X_\lambda, g_{\lambda\ \alpha\beta})$. To verify that I is an extremum when the field equations are satisfied, we need to vary ε and $\sqrt{-g}$; and to find the change in ε, we will need to find the change in u^α.

Variations of the metric and fluid

First order departures from an initial configuration can be described in two ways. The Eulerian perturbations in the quantities $Q(\lambda)$ are defined by

$$\delta Q = \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0} \tag{2.41}$$

and compare values of Q at the same point of the spacetime. In the region occupied by the original fluid, one can also introduce the Lagrangian perturbations

$$\Delta Q = \frac{d}{d\lambda}[X_{-\lambda}Q(\lambda)]|_{\lambda=0} \tag{2.42}$$

$$= (\delta + \pounds_\xi)Q, \tag{2.43}$$

where ξ^α generates the family of diffeomorphisms X_λ. That is, the curve $\lambda \to X_\lambda(P)$ has tangent $\xi^\alpha(P)$ at the point P. The field ξ^α is termed a Lagrangian displacement and may be regarded as the connecting vector joining fluid elements in the unperturbed configuration to the corresponding elements in the perturbed spacetime.

The first order changes in the variables Q can be expressed in terms of the displacement ξ^α and the Eulerian change in the metric

$$h_{\alpha\beta} = \delta g_{\alpha\beta}. \tag{2.44}$$

In fact, we will see that perturbations of the fluid variables can all be written in terms of $\Delta g_{\alpha\beta}$,

$$\Delta g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \tag{2.45}$$

We begin with the change in the four-velocity u^α. Let $t \longrightarrow c(t)$ be the initial path of a fluid element, $c_\lambda = X_\lambda \circ c$ the new path. Because X_λ drags c to c_λ, the Lagrangian change in c and in its tangent vector vanishes. That is, if w^α is tangent to c, then $w^\alpha_\lambda = X_\lambda w$ is tangent to c_λ. Thus $X_{-\lambda} w^\alpha_\lambda = w^\alpha$, independent of λ, implying

$$\Delta w^\alpha = \partial_\lambda (X_{-\lambda} w^\alpha_\lambda) = \partial_\lambda w^\alpha = 0. \tag{2.46}$$

Now w^α will not, in general have norm -1; even if we choose t to be proper time along the original path, t will not be proper time along c_λ. As a result, the Lagrangian change in the four-velocity is nonzero, depending on the change in the metric along the fluid trajectory, $\Delta g_{\alpha\beta} u^\alpha u^\beta$. We have

$$u^\alpha = \frac{w^\alpha}{(-w^\beta w_\beta)^{1/2}} = \frac{w^\alpha}{(-g_{\beta\gamma} w^\beta w^\gamma)^{1/2}};$$

$$\Delta u^\alpha = -\frac{1}{2}\frac{w^\alpha}{(-w_\delta w^\delta)^{3/2}}(-\Delta g_{\beta\gamma}w^\beta w^\gamma) \quad (2.47)$$

$$= \frac{1}{2}u^\alpha u^\beta u^\gamma \Delta g_{\beta\gamma}. \quad (2.48)$$

The change in baryon density can similarly be written in terms of $\Delta g_{\alpha\beta}$, because the number of baryons in a fluid element is conserved, and baryon conservation allows us to relate the change in baryon density to the change in a volume V orthogonal to u^α. The rest mass of baryons in V is

$$M_0 = \int_V \rho\sqrt{^3q}, \quad (2.49)$$

with ^3q the determinant of the metric q_{ab} orthgonal to u^α.

$$^3q = det q_{..} \quad (2.50)$$

From $N(\lambda) = N$, we have

$$0 = \frac{d}{d\lambda} \int_{V_\lambda} \rho(\lambda) \sqrt{q}(\lambda) = \int_V \Delta[\rho \sqrt{q}], \tag{2.51}$$

implying

$$\Delta(\rho \sqrt{q}) = 0.$$

Now the volume of a fluid element perpendicular to u^α is proportional to $\sqrt{q}$, and the fractional change in its volume is

$$\frac{\Delta V}{V} = \frac{\Delta \sqrt{q}}{\sqrt{q}}.$$

Recall, for any matrix $M(\lambda)$,

$$\frac{d}{d\lambda} \det M(\lambda) = \det M(\lambda)\ Tr \frac{d}{d\lambda} M(\lambda).$$

Then

$$\frac{\Delta\, q}{q} = \ q^{ab} \Delta q_{ab}.$$

Because $q_{\alpha\beta}$ is the projection operator onto a subspace orthogonal to u^α, and q_{ab} is the restriction of $q_{\alpha\beta}$ to that subspace, $q^{ab}\Delta q_{ab} = q^{\alpha\beta}\Delta q_{\alpha\beta} = q^{\alpha\beta}\Delta g_{\alpha\beta}$.

$$\frac{\Delta\, q}{q} = q^{\alpha\beta}\Delta g_{\alpha\beta};$$

and $\Delta(\rho\sqrt{{}^3g}) = 0$ implies

$$\frac{\Delta\rho}{\rho} = -\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta} \tag{2.52}$$

The equation means that the fractional increase in ρ is equal to the fractional decrease in the volume orthogonal to the 4-velocity. Next, to find the change $\Delta\varepsilon$ in the energy density, we use the energy conservation equation:

$$\begin{aligned} 0 = \Delta(u_\alpha\nabla_\beta T^{\alpha\beta}) &= -\Delta[(\varepsilon+p)\nabla_\beta u^\beta + u^\beta\nabla_\beta\varepsilon] && (2.53)\\ &= -u^\beta\nabla_\beta[\Delta\varepsilon + \frac{1}{2}(\varepsilon+p)q^{\alpha\beta}\Delta g_{\alpha\beta}] && (2.54)\end{aligned}$$

with first integral

$$\Delta\varepsilon = -\frac{1}{2}(\varepsilon + p)q^{\alpha\beta}\Delta g_{\alpha\beta}. \tag{2.55}$$

This expresses the fact that the flow is isentropic. In terms of the comoving rest-mass density, ρ, the first law of thermodynamics implies

$$\frac{\Delta\varepsilon}{\varepsilon + p} = \frac{\Delta\rho}{\rho}, \tag{2.56}$$

equivalent to (2.55), with $\frac{\Delta\rho}{\rho}$ given by (2.52).
The Lagrangian change in the pressure is similarly given by

$$\Delta p = \gamma p \frac{\Delta\varepsilon}{\varepsilon + p} = -\frac{1}{2}\gamma p\, q^{\alpha\beta}\,\Delta g_{\alpha\beta}, \tag{2.57}$$

where the adiabatic index γ is defined by

$$\gamma = \frac{\partial \log p(\rho, s)}{\partial \log \rho} = \frac{\varepsilon + p}{p}\frac{\partial}{\partial\varepsilon} p(\varepsilon, s). \tag{2.58}$$

Variation of the action

Using, as usual, the relation

$$\frac{d}{d\lambda}\int_{V_\lambda} f\varepsilon = \int_V \pounds_\xi(f\varepsilon) = \int_V \pounds_\xi(f\sqrt{-g})d^4x,$$

we have

$$\delta I_{\text{fluid}} = -\int \Delta(\varepsilon\sqrt{-g})d^4x.$$

Now

$$\frac{1}{\sqrt{-g}}\Delta\sqrt{-g} = \frac{1}{2}g^{\alpha\beta}\Delta g_{\alpha\beta} \;\; (\text{or} \;\; \Delta\varepsilon_{\gamma\delta\zeta\eta} = \varepsilon_{\gamma\delta\zeta\eta}\frac{1}{2}g^{\alpha\beta}\Delta g_{\alpha\beta} \;\;),$$

implying

$$\begin{aligned}\frac{1}{\sqrt{-g}}\Delta\left(\varepsilon\sqrt{-g}\right) &= -\frac{1}{2}[(\varepsilon+p)q^{\alpha\beta} - \varepsilon g^{\alpha\beta}]\Delta g_{\alpha\beta} - \frac{1}{2}[\varepsilon u^\alpha u^\beta + p q^{\alpha\beta}]\Delta g_{\alpha\beta} \\ &= -\frac{1}{2}T^{\alpha\beta}\Delta g_{\alpha\beta}. \qquad (2.59)\end{aligned}$$

Then

$$\delta T_{\text{fluid}} = \int \frac{1}{2} T^{\alpha\beta} \Delta g_{\alpha\beta} \sqrt{-g}\, d^4x = \int \frac{1}{2} T^{\alpha\beta} (h_{\alpha\beta} + 2\nabla_\alpha \xi_\beta) \sqrt{-g}\, d^4x$$
$$= \int (\frac{1}{2} T^{\alpha\beta} h_{\alpha\beta} - \nabla_\alpha T^{\alpha\beta} \xi_\beta) \sqrt{-g}\, d^4x. \qquad (2.60)$$

Requiring that $\delta I_{\text{fluid}} = 0$ for all ξ gives the equations of motion,

$$\nabla_\alpha T^{\alpha\beta} = 0. \qquad (2.61)$$

Requiring that $\delta I_{\text{GR}} + I_{\text{fluid}} = 0$ for all $h_{\alpha\beta}$ with

$$I_{\text{GR}} = \frac{1}{16\pi} \int R \sqrt{-g}\, d^4x, \qquad (2.62)$$

gives the field equations,

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}. \qquad (2.63)$$

Here we use

$$\delta I_{\text{GR}} = \frac{1}{16\pi} \int (-G^{\alpha\beta}) h_{\alpha\beta} \sqrt{-g}\, d^4x. \qquad (2.64)$$

Note that to obtain the field equations from an action for matter + gravity, one must define the energy momentum tensor by

$$T^{\alpha\beta} = 2\frac{\delta I_{\mathrm{matter}}}{\delta g_{\alpha\beta}}. \tag{2.65}$$