Project 1: The Wave Equation on the Schwarzschild Background in Eddington-Finkelstein Coordinates

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1 Introduction

This project investigates the time evolution of a massless scalar field on a Schwarzschild background by first deriving the relevant equations of motion and then writing an RNPL [1] code to investigating the scattering behaviour of spherically symmetric pulses of in-falling scalar radiation. For the purposes of the project, the mass of the scalar field is considered negligible compared to the mass of the black hole and so the back reaction of the scalar field on the metric is not considered.

1.1 The Wave Equation for a General, Static, Spherically Symmetric Metric

In the 3+1 formalism, the Schwarzschild metric can be written as:

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

where $\alpha \equiv \alpha(r)$ is the lapse function and $\beta \equiv \beta(r)$ is the shift vector which describes the velocity of the coordinate system as seen by Euleran observers. As we are in spherical symmetry, $\beta^i = (\beta, 0, 0)$, $\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0)$, where $\gamma_{ij} = \mathrm{diag}(a^2, r^2, r^2\sin^2\theta)$

The massless Klein-Gordon equation serves as our wave equation:

$$\nabla^a\nabla_a\phi(r,t) = 0 \qquad (2)$$

Expanding the derivative operators and simplifying, the wave equation may be written as a set of coupled first order equations [2]:

$$
\begin{aligned}
\partial_t \Phi &= \partial_r \left(\beta\Phi + \frac{\alpha}{a}\Pi \right) && (3)\\
\partial_t \Pi &= \frac{1}{r^2}\partial_r \left(r^2 \left(\beta\Pi + \frac{\alpha}{a}\Phi \right) \right) && (4)\\
\Phi &= \partial_r \phi && (5)\\
\Pi &= \frac{a}{\alpha}\left(\partial_t\phi - \beta\partial_r\phi \right) && (6)
\end{aligned}
$$

1.2 The Schwarzschild Solution in Ingoing Eddington-Finkelstein Coordinates

The usual form of the Schwarzschild metric is poorly behaved at the black hole horizon making it an unsuitable choice for our simulation. instead, we define the Regge-Wheeler tortoise coordinate $r_\star$ and a pair of ingoing and outgoing null coordinates u and v [2]:

$$
\begin{aligned}
r_\star &= r + 2M\ln\left(\frac{r}{2M} - 1\right) && (7)\\
u &= t - r_\star && (8)\\
v &= t + r_\star && (9)
\end{aligned}
$$

If we adopt a time-like coordinate $\tilde{t}$ based on the ingoing null coordinate v [2]:

$$
\tilde{t} = v - r = t + 2M\ln\left(\frac{r}{2M} - 1\right) \tag{10}
$$

Then the Schwarzschild metric takes the ingoing Eddington-Finkelstein (IEF) form and is well behaved at the event horizon ($r = 2M$) and for $r > 2M$:

$$
\begin{aligned}
ds^2 &= -\alpha^2 d\tilde{t}^2 + \beta a^2 dr d\tilde{t} + a^2 dr^2 + r^2 d\Omega^2 && (11)\\
\alpha &= \left(\frac{r}{r+2M}\right)^{1/2} && (12)\\
a &= \left(\frac{r}{r+2M}\right)^{-1/2} && (13)\\
\beta &= \frac{2M}{r+2M} && (14)
\end{aligned}
$$

1.3 Radiation Boundary Conditions

As $r \to \infty$ the metric presented above approaches the Minkowskii metric in spherical coordinates:

$$
ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{15}
$$

In this limit, the components of the metric become:

$$
\begin{aligned}
\alpha &= 1 && (16)\\
a &= 1 && (17)\\
\beta &= 0 && (18)\\
& && (19)
\end{aligned}
$$

and the wave equation simplifies as follows:

$$
\begin{aligned}
\nabla^a \nabla_a \phi(r,t) &= 0 && (20)\\
\frac{1}{-g}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g} g^{\mu\nu}\frac{\partial \phi}{\partial x^\mu}\right) &= 0 && (21)\\
\frac{1}{r^2 \sin}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r}\right) - \frac{\partial}{\partial t}\left(r^2 \sin\theta \frac{\partial \phi}{\partial t}\right)\right) &= 0 && (22)\\
\partial_{tt}(r\phi) - \partial_{rr}(r\phi) &= 0 && (23)
\end{aligned}
$$

We do not want interference originating outside the domain of integration, so we look to impose outgoing radiation or Sommerfeld boundary conditions. The outgoing solution of the asymptotic wave equation is:

$$r\phi = g(t-r) \tag{24}$$

which has the property:

$$\partial_t (r\phi) + \partial_r (r\phi) = 0 \tag{25}$$

Therefore, we can apply this condition at the boundary and so long as our domain is large enough that the Minkowskii space metric is a fair approximation, we will have approximate outgoing radial boundary conditions.

Noting that our wave equation is massless and must propagate at the speed of light, we can examine the curved space metric and find the radial null characteristics:

$$
\begin{aligned}
0 &= (-\alpha^2 + a^2\beta^2) + 2a^2\beta\frac{dr}{dt} + a^2\left(\frac{dr}{dt}\right)^2 && (26)\\
\left(\frac{dr}{dt}\right)_\pm &= \left(-2a^2\beta \pm \sqrt{4a^4\beta^2 - 4\left(-\alpha^2 + a^2\beta^2\right)}\right)/(2a^2) && (27)\\
\left(\frac{dr}{dt}\right)_\pm &= -\beta \pm \frac{\alpha}{a} = c_\pm && (28)
\end{aligned}
$$

Substituting for the wave speed at the boundary gives us a better approximation in slightly curved space [2]:

$$
\begin{aligned}
r\phi &= g(c_+ t - r) && (29)\\
\partial_t (r\phi) + \left(-\beta + \frac{\alpha}{a}\right)\partial_r (r\phi) &= 0 && (30)
\end{aligned}
$$

1.4 Conserved Mass for the Model

As we are interested in determining the fraction of the mass which is scattered by the black hole, we must find a conserved mass like quantity. By contracting the stress energy tensor, T^{ab}, for the scalar field with t^a, a time like killing vector field, gives us a conserved energy-momentum 4-vector, J^a [2]:

$$\begin{aligned} J^a &= T^{ab}t_b = \left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right)t_b && (31)\\ \nabla_a J^a &= \nabla_a\left(T^{ab}t_b\right) = \left(\nabla_a T^{ab}\right)t_b + T^{ab}\nabla_a t_b = \left(\nabla_a T^{ab}\right)t_b + T^{ab}\nabla_{(a}t_{b)} = 0 && (32)\end{aligned}$$

Where the last line is the result of the conservation and symmetry of T^{ab} and the Killing equations. We can then integrate $\nabla_a J^a$ over a spacetime volume and apply Gauss's theorem:

$$\int_V \nabla_a J^a = \int_{\partial V} N_a J^a \tag{33}$$

where ∂V is the boundary of the integration region and N^a is the normal vector to ∂V. Now assuming the flux is 0 at spatial infinity, we can take out region to the the 4-volume bounded between any two hypersurfaces Σ_t and $\Sigma_{t'}$ [2]:

$$\begin{aligned} 0 &= \int_{\Sigma_t} N_a J^a - \int_{\Sigma_t'} N_a J^a && (34)\\ m_\infty &= \int_{\Sigma_t} N_a J^a && (35)\\ m_\infty &= \int \left(-\alpha T^t_t\right)\left(ar^2\sin^2\theta\right)drd\phi = 4\pi\int -r^2\alpha a T^t_t dr && (36)\end{aligned}$$

In the above expression, we have made use of the identities $t^a = (1,0,0,0)$ and $N_a = (-\alpha, 0, 0, 0)$. In terms of the variables Π and Φ, the conserved mass for the system may be written [2]:

$$m_r = \int_{2M}^{r} -4\pi\tilde{r}^2\alpha a T^t_t d\tilde{r} \tag{37}$$

1.5 Initial Data

Although you could study scattering behaviour for nearly any initial pulse configuration, we restrict ourselves to Gaussian pulses:

$$\phi(r, t_0) = A\exp\left(-\left(\frac{r - r_0}{\Delta}\right)^2\right) \tag{38}$$

A quick inspection of our outgoing boundary condition reveals that for initially ingoing initial data we require:

$$\partial_t (r\phi) + \left(-\beta - \frac{\alpha}{a}\right) \partial_r (r\phi) = 0 \tag{39}$$

1.6 Methodology

The program was written in RNPL using a simple Crank-Nicholson difference scheme with $O(h^2)$ centered finite difference approximations for the spatial derivatives on the interior of the region and $O(h^2)$ forwards and backwards FDA's at the boundaries of the region. As the outgoing radial characteristic at $r = 2M$ is null, there was no need to impose boundary conditions at this point. Instead, the evolution equation was used with all centered derivatives replaced with forward approximations. As discussed above, Sommerfeld conditions were implemented on the outer boundary.

The simulation was run for $M = 1$, $R = 200$, $A = 1$ and $r_0 = 50$ a variety of width parameters Δ and the ratio of the scattered mass to the initially ingoing mass was computed. The results are shown below.

2 Results

Despite the approximately ingoing boundary conditions, a non negligible fraction of the radiation is initially outgoing. The ratio of the scattered mass to the initial mass is plotted below in Figure 1. The final mass was computed by allowing the center of the reflected mass to travel to approximately $R = 50$. Figure 2 shows the evolution of a narrow (nearly entirely absorbed) pulse. Videos of the evolution of the scalar field may be found on my website.

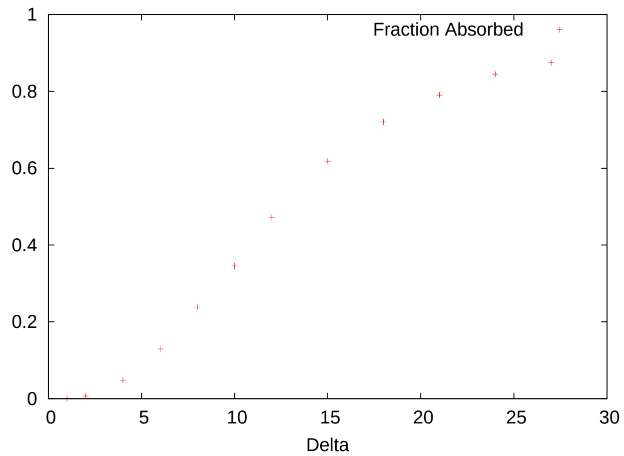

Figure 1: Fraction of mass scattered by the black hole.

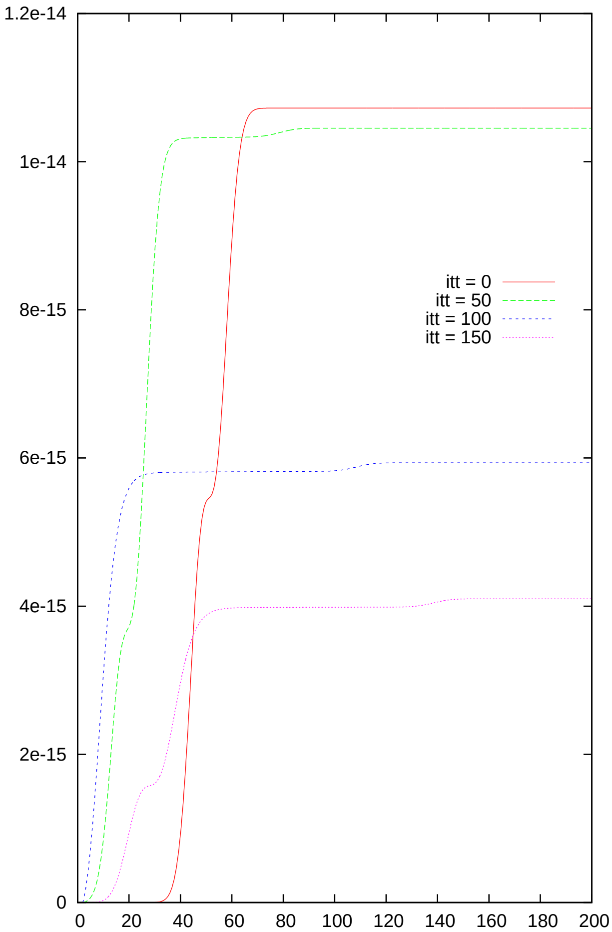

Figure 2: Evolution of a narrow pulse ($\Delta = 4$).

References

[1] M. C. Robert Marsa, “The rnpl reference manual,” http://laplace.physics.ubc.ca/People/marsa/rnpl/refman/refman.html, 1995.

[2] M. Choptuik, “Project 1: The wave equation on the schwarzschild background in eddington-finkelstein coordinates,” http://laplace.physics.ubc.ca/People/matt/Teaching/03Vancouver/p1.pdf, 2003.