A Numerical Study of Boson Star Binaries

Bruno C. Mundim
Department of Physics and Astronomy
University of British Columbia
SCAIM - 2 December, 2008

Outline

- General Motivation
- GR in a Nutshell
- 3+1 Formalism
- Matter Model: Scalar Field
- Conformal Flat Approximation (CFA)
  - Motivation
  - Representative work
  - Formalism and Equations of motion
  - Discretization Scheme and Numerical Techniques
  - Compactification of the spatial domain: challenges
- Current Project: Coalescence of Boson Stars
  - Motivation
  - Questions to be addressed
- Results and Future Directions
- Appendix: Boson Stars in Spherical Symmetry

General Motivation

- Why study compact binaries?
  - One of most promising sources of gravitational waves
  - It is a good laboratory to study the phenomenology of strong gravitational fields

- Why boson stars?
  - Plunge and merge phase of the inspiral of compact objects is characterized by a strong dynamical gravitational field. In this regime gross features of fluid and boson stars' dynamics may be similar
  - Since the details of the dynamics of the stars (e.g. shocks) tend not to be important gravitationally, boson star binaries may provide some insight into NS binaries

- Development of a computational infrastructure for 3D codes
  - 3D numerical relativistic calculations are computationally very expensive. Need for more efficient computational techniques: AMR, parallelization.
    - This infrastructure has been constructed by Frans Pretorius: PAMR

GR in a nutshell

The notion of distance between two points in a *Riemannian* manifold or the interval between two events in a *Lorentzian* manifold is encoded on the metric tensor:

$$g \equiv ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

Examples - Vacuum spacetimes:

- Minkowski/Flat spacetime in cartesian coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

- Schwarzschild spacetime in spherical coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$

GR in a nutshell

- The notion of force (Newtonian gravity) is replaced by the notion of curvature.
- Matter/Energy is responsible for spacetime curvature. In Wheeler's words:
  - "Matter tells spacetime how to curve and spacetime tells matter how to move."
- Einstein Field Equations:

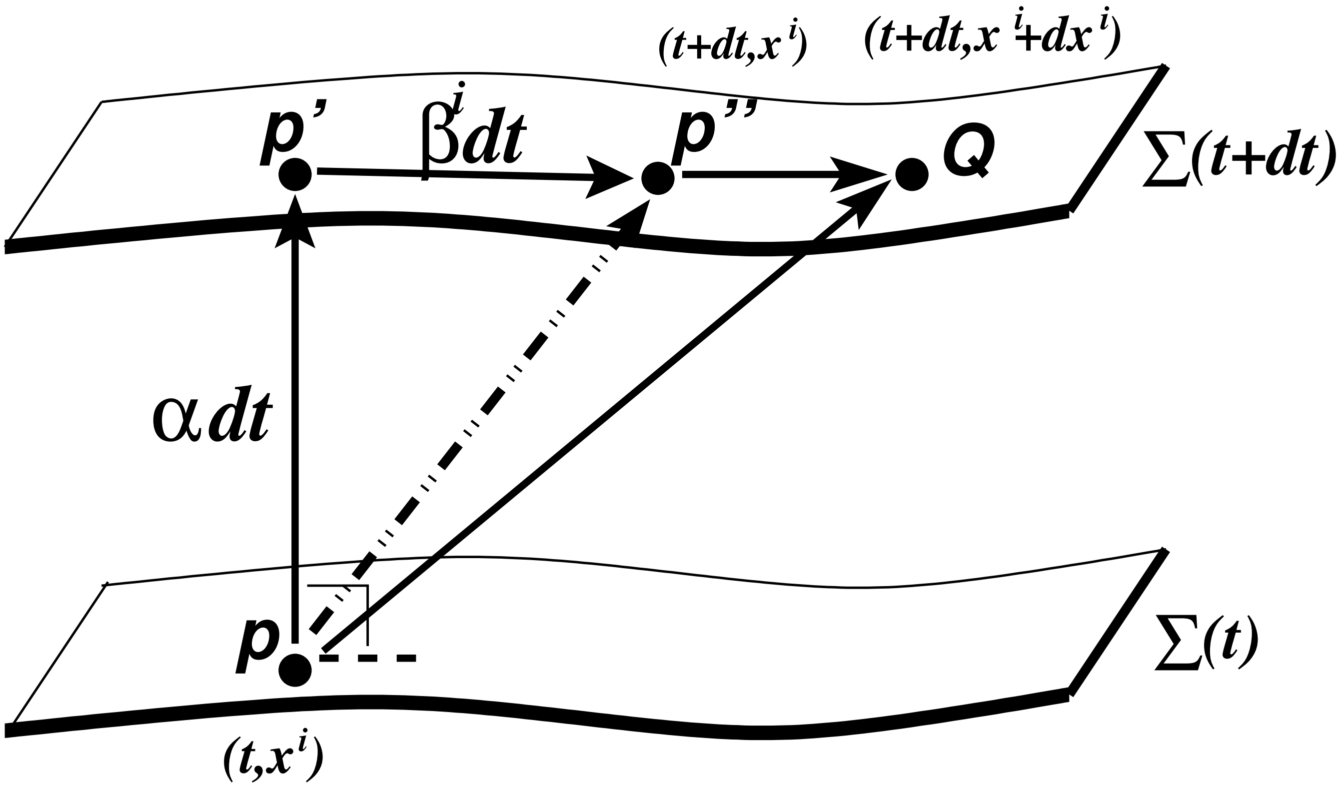
$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

  - System of non-linear, time-dependent, partial differential equations
  - No analytic solution except in special cases
  - Solution for most relevant astrophysical scenarios must be constructed numerically
  - Its tensorial nature gives rise to several different formalisms
  - ADM / $3+1$ formalism: slice spacetime in spacelike hypersurfaces; use Einstein equations to evolve in time the 3-geometry of an initial hypersurface in order to construct the spacetime (i.e. the 4-dimensional metric, $g_{\mu\nu}$)

3+1 formalism

- 3+1 line element

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$



A schematic representation of the ADM (or 3+1) decomposition

3+1 formalism

- Constraint Equations: From $G_{0i} = 8\pi T_{0i}$, which do not contain 2nd time derivatives of the γ_{ij}

- Hamiltonian Constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{4}$$

where R is the 3-dim. Ricci scalar, and $K \equiv K^i{}_i$ is the mean extrinsic curvature.

- Momentum Constraint

$$D_i K^{ij} - D^j K = 8\pi j^i \tag{5}$$

- Evolution Equations: From definition of extrinsic curvature, $G_{ij} = 8\pi T_{ij}$, and Ricci's equation.

$$\mathcal{L}_t \gamma_{ij} = \mathcal{L}_\beta \gamma_{ij} - 2\alpha K_{ij} \tag{6}$$

$$\mathcal{L}_t K_{ij} = \mathcal{L}_\beta K_{ij} - D_i D_j \alpha + \alpha\left(R_{ij} + K K_{ij} - 2K_{ik}K^k{}_j\right) - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \tag{7}$$

Matter Model: Scalar Field

- Star-like solutions: A massive complex field is chosen as matter source because it is a simple type of matter that allows a star-like solution and because there will be no problems with shocks, low density regions, ultrarelativistic flows, etc in the evolution of this kind of matter as opposed to fluids

- Static spacetimes: *Complex* scalar fields allow the construction of static spacetimes in opposition to *real* scalar fields. The matter content is then described by:

$$\Phi = \phi_1 + i\phi_2 \tag{8}$$

where ϕ_1 and ϕ_2 are real-valued

- The Lagrangian density associated with this field is given by:

$$L_\Phi = -\frac{1}{8\pi}(g^{ab}\nabla_a\Phi\nabla_b\Phi^* + m^2\Phi\Phi^*) \tag{9}$$

- Extremizing this action with respect to each component of the scalar field, we get the Klein-Gordon equation

$$\Box\phi_A - m^2\phi_A = 0 \qquad A = 1, 2 \tag{10}$$

Matter Model: Scalar Field

- From the point of view of ADM formalism the Hamiltonian formulation of the dynamics of scalar field is more useful

- The conjugate momentum field is defined as

$$\Pi_A \equiv \frac{\delta(\sqrt{-g} L_{\phi_A})}{\delta \dot{\phi}_A} \tag{11}$$

- In terms of these fields, the dynamical equations are given by

$$\partial_t \phi_A = \frac{\alpha^2}{\sqrt{-g}} \Pi_A + \beta^i \partial_i \phi_A \tag{12}$$

$$\partial_t \Pi_A = \partial_i(\beta^i \Pi_A) + \partial_i(\sqrt{-g}\gamma^{ij}\partial_j \phi_A) - \sqrt{-g} m^2 \phi_A \tag{13}$$

Matter Model: Scalar Field

- The stress-energy tensor is given by

$$T_{ab} = -2\frac{\delta L_\Phi}{\delta g^{ab}} + g_{ab}L_\Phi \tag{14}$$

- We have the following ADM components of the stress tensor

$$\begin{aligned}
\rho &= n^\mu n^\nu T_{\mu\nu} = \frac{1}{8\pi}\sum_{A=1}^{2}\left(\frac{\alpha^2}{(-g)}\Pi_A^2 + \gamma^{ij}\partial_i\phi_A\partial_j\phi_A + m^2\phi_A^2\right) \\
j^i &= -n^\mu T_\mu{}^i = \frac{1}{8\pi}\sum_{A=1}^{2}\left(-2\frac{\alpha\Pi_A}{\sqrt{-g}}\gamma^{ij}\partial_j\phi_A\right) \\
S_{ij} &= T_{ij} \\
&= \frac{1}{8\pi}\sum_{A=1}^{2}\left(2\partial_i\phi_A\partial_j\phi_A + \gamma_{ij}\left[\frac{\alpha^2\Pi_A^2}{(-g)} - \gamma^{mn}\partial_m\phi_A\partial_n\phi_A - m^2\phi_A^2\right]\right)
\end{aligned} \tag{15}$$

Conformal Flat Approximation (CFA)

- Motivation
  - Facts and assumptions:
    - Full 3D Einstein equations are very complex and computationally expensive to solve
    - Heuristic assumption that the dynamical degrees of freedom of the gravitational fields, i.e. the gravitational radiation, play a small role in at least some phases of the strong field interaction of a merging binary
    - Gravitational radiation is small in most systems studied so far
  - An approximation candidate:
    - CFA effectively eliminates the two dynamical degrees of freedom, simplifies the equations and allows a fully constrained evolution
    - CFA allows us to investigate the same kind of phenomena observed in the full relativistic case, such as the description of compact objects and the dynamics of their interaction; black hole formation; critical phenomena
    - CFA has been used in the past with promising results in certain cases (Wilson-Matthews studies of coalescence of neutron stars; Bruno Rousseau's master's thesis)

Representative Work

- Wilson, Matthews, Marronetti, Phys. Rev. D 54, 1317 (1996)
  - Study of general relativistic hydrodynamics of a coalescing neutron-star binary system
  - They discuss the evidence that, for a realistic neutron-star equation of state, general relativistic effects may cause the stars to individually collapse into black holes prior to merging
  - Strong fields cause the last stable orbit (ISCO) to occur at a larger separation distance and lower frequency than previously estimated.
- E. Flanagan, Phys. Rev. Lett. 82, 1354 (1999): inconsistency in the solution of the shift vector.
- Matthews, Wilson, gr-qc/9911047 (1999): Incorporation of correction: compression effect still present but smaller for some angular momentum. Orbital frequency increases towards that expected from Post-Newtonian solutions.
- Bruno Rousseau' masters thesis Boson stars studied in axisymmetry under conformally flat approximation have been shown to behave similarly to the spherical solutions of the Einstein-Klein-Gordon equations under small perturbation

Conformal Flat Approximation (CFA)

- Formalism
  - The CFA prescribes a conformally flat spatial metric at all times
  - Introduce a flat metric f_{ij} as a base / background metric:

$$\gamma_{ij} = \psi^4 f_{ij} \tag{16}$$

  where the conformal factor ψ is a positive scalar function describing the ratio between the scale of distance in the curved space and flat space($f_{ij} \equiv \delta_{ij}$ in cartesian coordinates)

  - In this approximation all of the geometric variables can be computed from the constraints as well as from a specific choice of coordinates
  - Maximum slicing condition is used to fix the time coordinate

$$K^i_i = 0$$
$$\partial_t K^i_i = 0 \tag{17}$$

Conformal Flat Approximation (CFA)

- Slicing Condition
  - Gives an elliptic equation for the lapse function α

$$\nabla^2\alpha = -\frac{2}{\psi}\vec{\nabla}\psi \cdot \vec{\nabla}\alpha + \alpha\psi^4 \left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right) \tag{18}$$

- Hamiltonian Constraint
  - Gives an elliptic equation for the conformal factor ψ

$$\nabla^2\psi = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right) \tag{19}$$

- Momentum Constraints
  - Given elliptic equations for the shift vector components β^i

$$\begin{aligned}\nabla^2\beta^j \quad = \quad & -\frac{1}{3}\hat{\gamma}^{ij}\partial_i\left(\vec{\nabla}\cdot\vec{\beta}\right) + \alpha\psi^4 16\pi J^j - \partial_i\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\hat{\gamma}^{ik}\partial_k\beta^j\right. \\ & \left. + \hat{\gamma}^{jk}\partial_k\beta^i - \frac{2}{3}\hat{\gamma}^{ij}\left(\vec{\nabla}\cdot\vec{\beta}\right)\right] \end{aligned} \tag{20}$$

Conformal Flat Approximation (CFA)

- Note that $K_{ij}K^{ij}$ can also be expressed in terms of the flat operators. It ends up being expressed as flat derivatives of the shift vector:

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2}\left(\hat{\gamma}_{kn}\hat{\gamma}^{ml}\hat{D}_m\beta^k\hat{D}_l\beta^n + \hat{D}_m\beta^l\hat{D}_l\beta^m - \frac{2}{3}\hat{D}_l\beta^l\hat{D}_k\beta^k\right) \qquad (21)$$

Conformal Flat Approximation (CFA)

- 3d Cartesian Coordinates

$$\partial_t \phi_A = \frac{\alpha}{\psi^6}\Pi_A + \beta^i \partial_i \phi_A \tag{22}$$

$$\begin{aligned} \partial_t \Pi_A &= \partial_x \left(\beta^x \Pi_A + \alpha \psi^2 \partial_x \phi_A\right) + \partial_y \left(\beta^y \Pi_A + \alpha \psi^2 \partial_y \phi_A\right) \\ &+ \partial_z \left(\beta^z \Pi_A + \alpha \psi^2 \partial_z \phi_A\right) - \alpha \psi^6 \frac{dU(\phi_0^2)}{d\phi_0^2}\phi_A \end{aligned} \tag{23}$$

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = -\frac{2}{\psi}\left[\frac{\partial \psi}{\partial x}\frac{\partial \alpha}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \alpha}{\partial y} + \frac{\partial \psi}{\partial z}\frac{\partial \alpha}{\partial z}\right] + \alpha \psi^4 \left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right) \tag{24}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right) \tag{25}$$

Conformal Flat Approximation (CFA)

- x component of the shift vector in cartesian coordinates

$$\frac{\partial^2\beta^x}{\partial x^2}+\frac{\partial^2\beta^x}{\partial y^2}+\frac{\partial^2\beta^x}{\partial z^2} = -\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^y}{\partial y}+\frac{\partial\beta^z}{\partial z}\right)+\alpha\psi^4\,16\pi J^x$$
$$-\frac{\partial}{\partial x}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}\frac{\partial\beta^x}{\partial x}-\frac{2}{3}\left(\frac{\partial\beta^y}{\partial y}+\frac{\partial\beta^z}{\partial z}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial y}+\frac{\partial\beta^y}{\partial x}\right]$$
$$-\frac{\partial}{\partial z}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial z}+\frac{\partial\beta^z}{\partial x}\right] \qquad (26)$$

- $K_{ij}K^{ij}$ in 3d cartesian coordinates

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2}\left[\left(\frac{\partial\beta^x}{\partial x}\right)^2+\left(\frac{\partial\beta^x}{\partial y}\right)^2+\left(\frac{\partial\beta^x}{\partial z}\right)^2+\left(\frac{\partial\beta^y}{\partial x}\right)^2+\left(\frac{\partial\beta^y}{\partial y}\right)^2+\left(\frac{\partial\beta^y}{\partial z}\right)^2\right.$$
$$+\left(\frac{\partial\beta^z}{\partial x}\right)^2+\left(\frac{\partial\beta^z}{\partial y}\right)^2+\left(\frac{\partial\beta^z}{\partial z}\right)^2+\frac{\partial}{\partial x}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^x$$
$$+\frac{\partial}{\partial y}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^y+\frac{\partial}{\partial z}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^z$$
$$\left.-\frac{2}{3}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}\right)^2\right] \qquad (27)$$

Conformal Flat Approximation (CFA)

- Then the following set of functions completely characterize the geometry at each time slice

$$\alpha = \alpha(t, \vec{r}), \quad \psi = \psi(t, \vec{r}), \quad \beta^i = \beta^i(t, \vec{r}) \tag{28}$$

  where $\vec{r}$ depends on the coordinate choice for the spatial hypersurface

- The solution of the gravitational system under CFA and maximal slicing condition can be summarized as:
  - Specify initial conditions for the complex scalar field
  - Solve the elliptic equations for the geometric quantities on the initial slice
  - Update the matter field values to the next slice using their equation of motion
  - For the new configuration of matter fields, re-solve the elliptic equations for the geometric variables and again allow the matter fields to react and evolve to the next slice and so on

Conformal Flat Approximation (CFA)

- Discretization Scheme:

$$Lu - f = 0 \qquad \Rightarrow \qquad L^h u^h - f^h = 0 \tag{29}$$

  - For hyperbolic operators L: second order accurate Crank-Nicholson scheme.
  - For elliptic operators L: second order accurate centred finite difference operators.
  - Dirichlet Boundary conditions applied.

- Numerical Techniques:

  - pointwise Newton-Gauss-Seidel (NGS) iterative technique was used to solve the finite difference equations originated from the hyperbolic set of equations.
  - Multigrid Full Approximation Storage Algorithm (FAS) was applied on the discrete version of the elliptic set of equations. NGS is used in this context as a smoother of the solution error.

Compactification of the spatial domain: challenges

- Definition: Compactification of the spatial domain means to map $\mathbb{R}$ into a finite subinterval $M \in \mathbb{R}$:

$$\xi : \mathbb{R} \longrightarrow [-1, 1] \tag{30}$$

- As this subinterval can be remapped in any other finite one, there is no loss of generality if $[-1, 1]$ interval is chosen.

- Then all that is left is to find a particular function $\xi = \xi(x) \in C^2$ to do this map. In our case we chose *hyperbolic tangent* as the compactification function for each spatial dimension:

$$\chi = \tanh(x) \tag{31}$$

$$\eta = \tanh(y) \tag{32}$$

$$\zeta = \tanh(z) \tag{33}$$

- Main advantage: The boundary conditions corresponding to asymptotically flat spacetime (AFS) can be set exactly, ie they are Dirichlet boundary conditions. For non-compact domain, the boundary conditions for AFS are set approximately as Robin boundary conditions.

Compactification of the spatial domain: challenges

- Main Disadvantage: The elliptical equations for the geometric quantities become anisotropic. For example, the hamiltonian constraint after compactification can be written as:

$$(1-\chi^2)^2\psi_{,\chi\chi} - 2\chi(1-\chi^2)\psi_{,\chi} + (1-\eta^2)^2\psi_{,\eta\eta} - 2\eta(1-\eta^2)\psi_{,\eta} + (1-\zeta^2)^2\psi_{,\zeta\zeta} - 2\zeta(1-\zeta^2)\psi_{,\zeta} = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right) \quad (34)$$

- The functions in front of the second order derivative terms may differ drastically from one region to the other in the compact space. This difference becomes bigger as we increase the resolution of the numerical solution.

- Multigrid solver: It is the most efficient method to solve numerically elliptical equations (*O(N)* where N is the number of unknowns). The heart of a good Multigrid solver is the relaxation method whose main function is to smooth the solution found on the finer grid.

- Anisotropic elliptic equations require more sophisticated smoothers. So far there is no parallel and AMR computation infrastructure capable of handling anisotropic elliptic equations. We had to postpone the solution of this problem for the moment and we will focus on the non-compact coordinates equations.

Current Project - Coalescence of Boson Stars

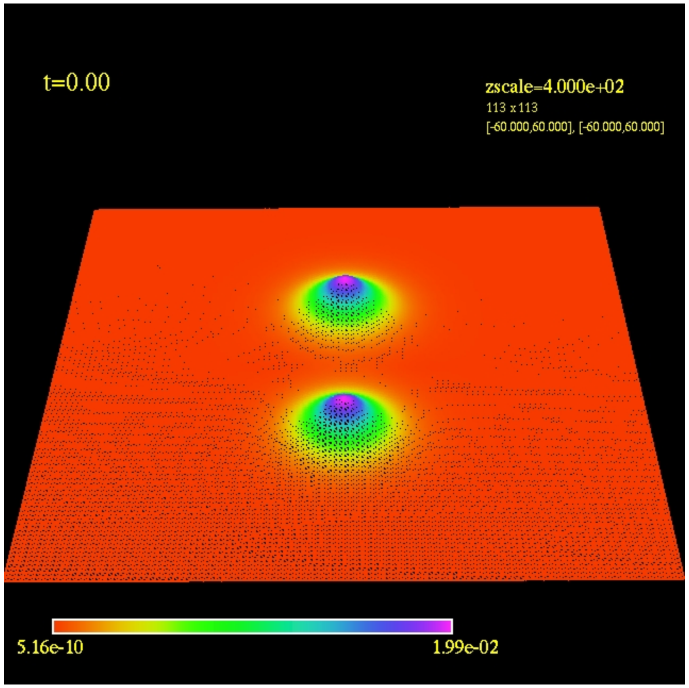
- Motivation
  - Controversy:
    - Wilson-Mathews compression effect results raised a controversy about the validity of the conformal flat approximation
    - In order to decide if CFA is a good approximation to model compact binaries it would be interesting to simulate it using a simpler model
  - Matter similarities:
    - Fluid stars and Boson stars have some similarity concerning the way they are modelled, e.g. both can be parametrized by their central density ρ_0 and have qualitatively similar plots of total mass vs ρ_0
    - Then in the strong field regime for the compact binary system the dynamics may not depend sensitively on the details of the model
    - Advantage of using scalar fields: no problems with shocks, evolution done by Klein-Gordon eqn, should not present any stability problem.

Current Project - Coalescence of Boson Stars

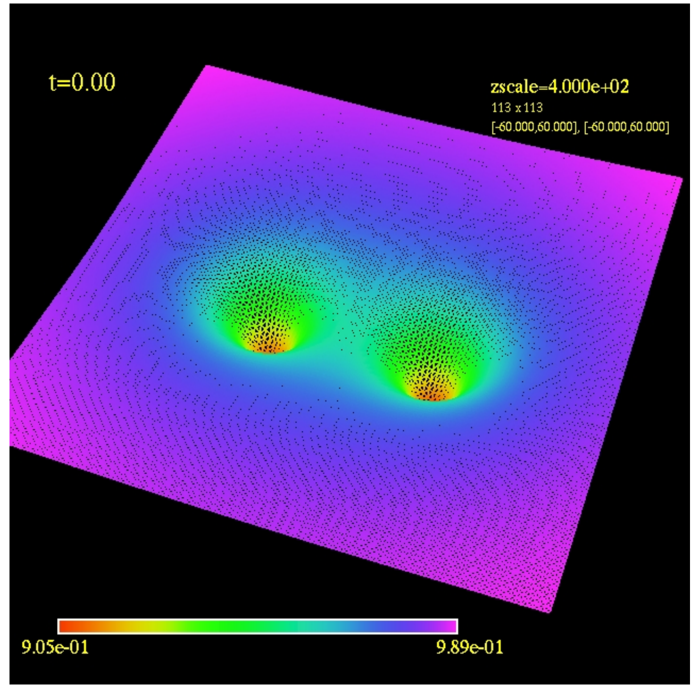
- Questions to be addressed
  - Would the individual collapse occur before merging for boson stars as well or it is model dependent?
  - How good is the approximation? How do we test if the results are close to solutions of Einstein equations?
  - Is the individual collapse a spurious result coming from CFA?
  - What is the final result of the merging? Can we compare to results from other techniques?
  - Where is the ISCO? Does this result match to the fluid star ones? Can be at least qualitatively compared?
  - How can we extract the gravitational waveforms from this system?

Results:

- Orbital Dynamics: 2 boson stars - $v_x = 0.09$



$Z = 0$ slice for $|\phi|$

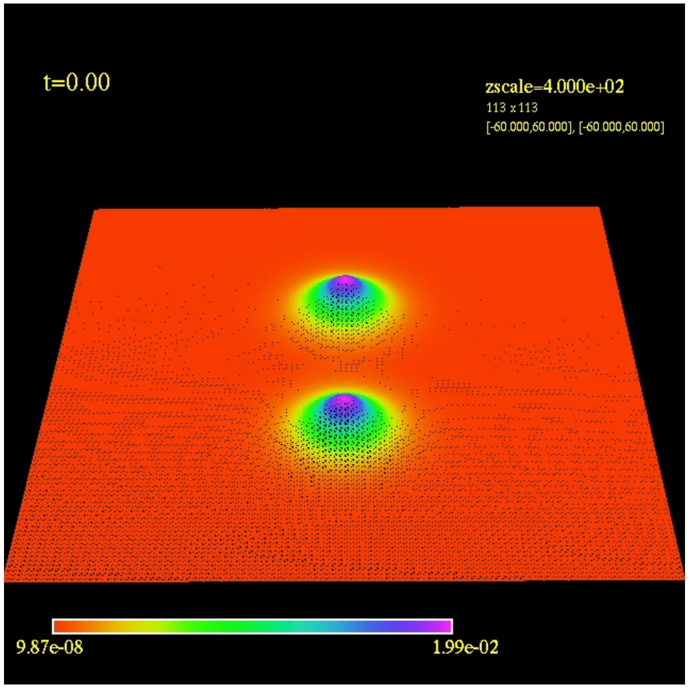


$Z = 0$ slice for α

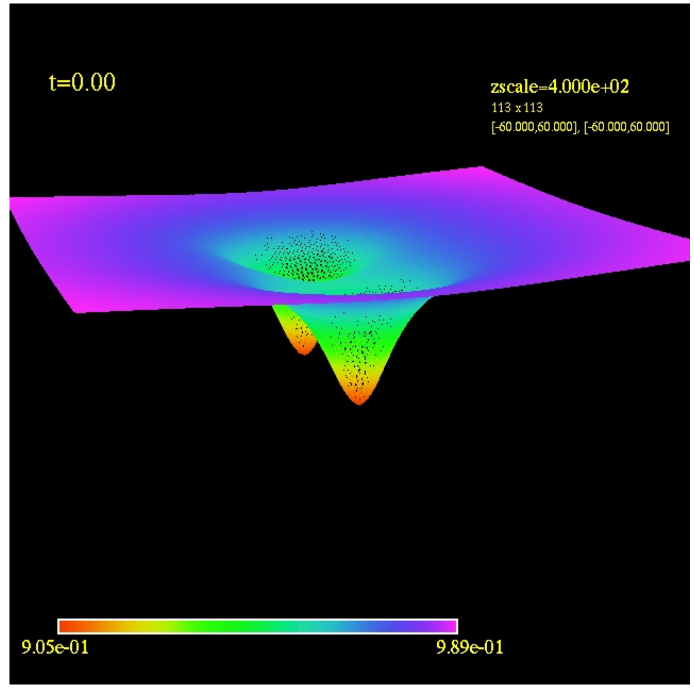
- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 4500$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 285 hours (12 days).

Results:

- Orbital Dynamics: 2 boson stars - $v_x = 0.07$



$Z = 0$ slice for $|\phi|$

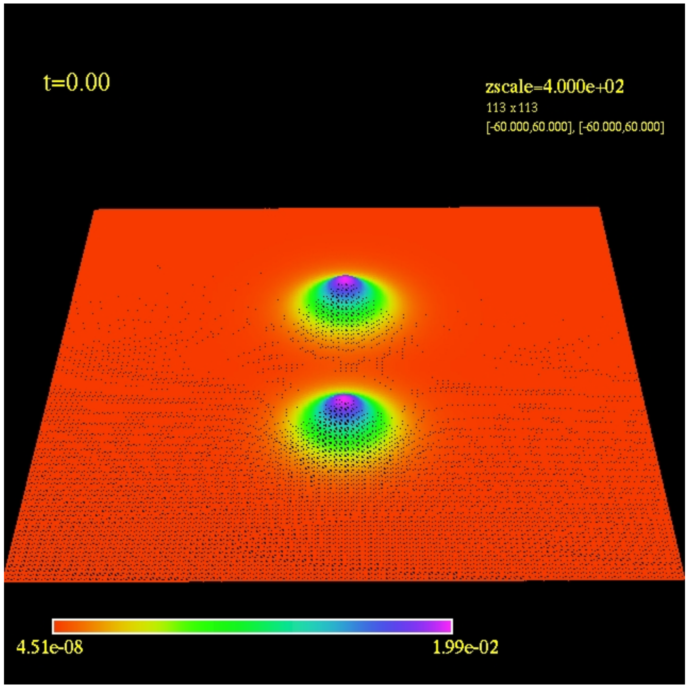


$Z = 0$ slice for α

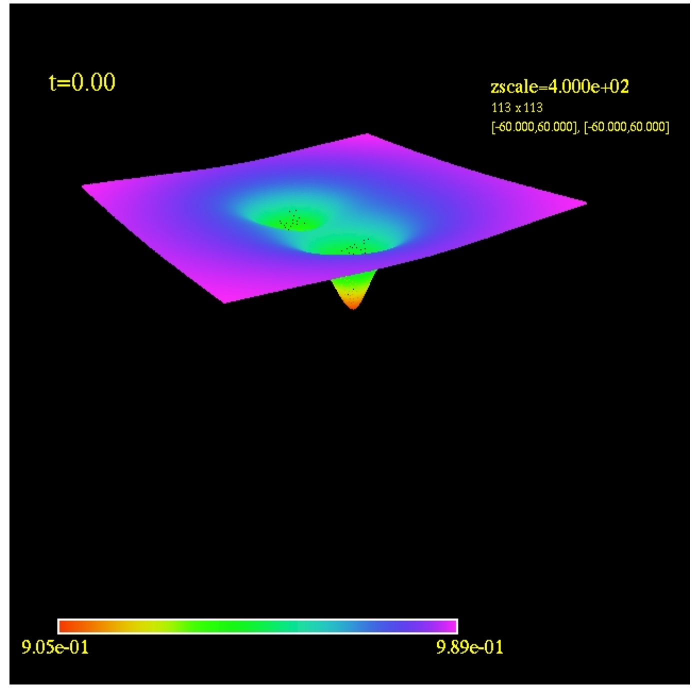
- ϕ_0 : 0.02. Physical coordinate domain: 120 per edge. Physical time: $t = 2250$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 158 hours (6.5 days).

Results:

- Orbital Dynamics: 2 boson stars - $v_x = 0.05$



$Z = 0$ slice for $|\phi|$

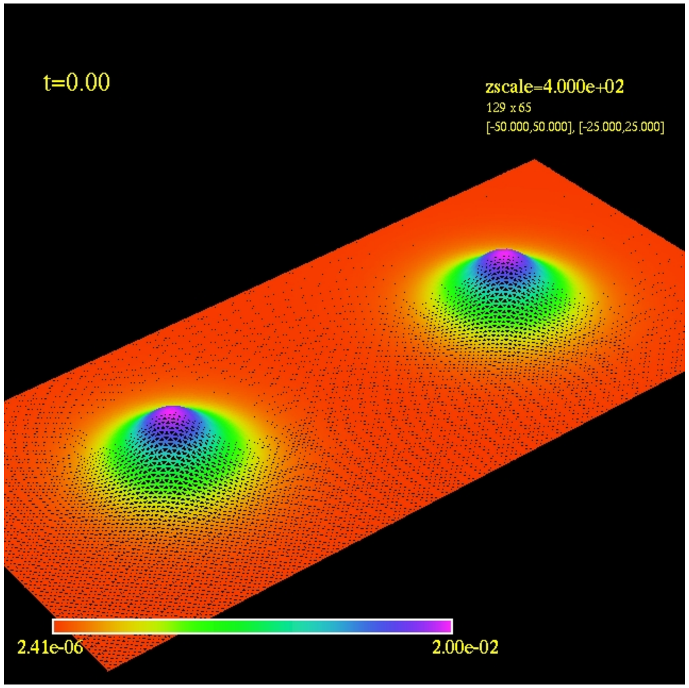


$Z = 0$ slice for α

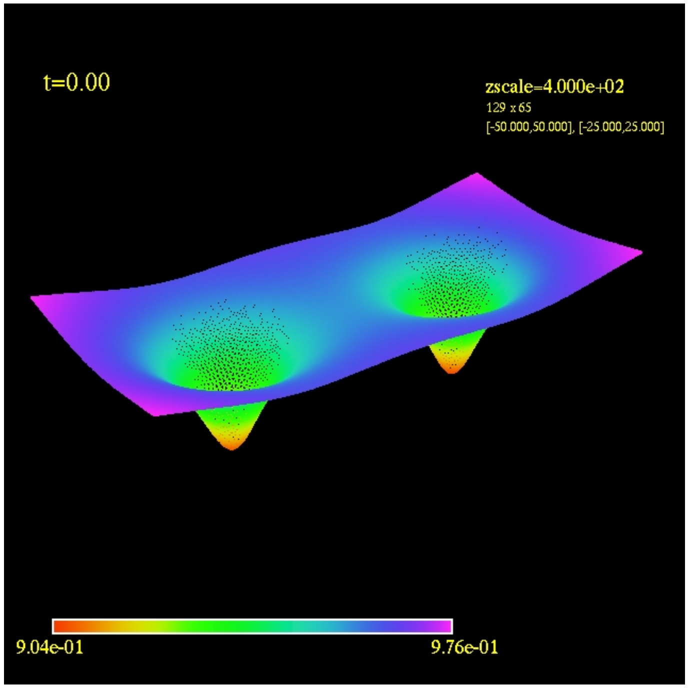
- ϕ_0 : 0.02. Physical coordinate domain: 120 per edge. Physical time: $t = 1500$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 115 hours (4.5 days).

Results:

- Head-on collision: 2 boson stars - $v_x = 0.4$



$Z = 0$ slice for $|\phi|$



$Z = 0$ slice for α

- $\phi_0 : 0.02$. Physical coordinate domain: $[-50, 50, -25, 25, -25, 25]$. Total physical time: $t = 140$. Simulation parameters: $\lambda = 0.4$; Grid size: $[N_x, N_y, N_z] = [129, 65, 65]$;

Future Directions:

- Code Improvements:
  - Add some features to the 3D code/equations such as:
    - An increase in resolution by implementing code parallelization and applying adaptive mesh refinement techniques (use of PAMR infrastructure)
    - The improvement of the boundary conditions in order to better capture the physical boundary conditions: asymptotically flat spacetime.
    - A radiation back reaction term to the Klein-Gordon equation in order to allow the effects of the radiation into the dynamics of the system.

Appendix: Boson Stars in Spherical Symmetry

- Spherically Symmetric Spacetime (SS):

$$ds^2 = \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta\, dtdr + a^2dr^2 + r^2b^2d\Omega^2\,, \tag{35}$$

- Hamiltonian constraint:

$$-\frac{2}{arb}\left\{\left[\frac{(rb)'}{a}\right]' + \frac{1}{rb}\left[\left(\frac{rb}{a}(rb)'\right)' - a\right]\right\} + 4K^r{}_r K^\theta{}_\theta + 2{K^\theta{}_\theta}^2 = 8\pi\left[\frac{|\Phi|^2 + |\Pi|^2}{a^2} + m^2|\phi|^2\right] \tag{36}$$

- Momentum constraint:

$$K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = \frac{2\pi}{a}\left(\Pi^*\Phi + \Pi\Phi^*\right)\,. \tag{37}$$

where the auxiliary field variables were defined as:

$$\Phi \equiv \phi'\,, \tag{38}$$

$$\Pi \equiv \frac{a}{\alpha}\left(\dot\phi - \beta\phi'\right)\,, \tag{39}$$

Boson Stars in Spherical Symmetry

- Evolution equations

$$\dot{a} = -\alpha a K^r{}_r + (a\beta)' \quad (40)$$

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' . \quad (41)$$

$$\dot{K^r{}_r} = \beta K^r{}'_r - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left\{-\frac{2}{arb}\left[\frac{(rb)'}{a}\right]' + KK^r{}_r - 4\pi\left[\frac{2|\Phi|^2}{a^2} + m^2|\phi|^2\right]\right\} \quad (42)$$

$$\dot{K^\theta{}_\theta} = \beta K^\theta{}'_\theta + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left[\frac{\alpha rb}{a}(rb)'\right]' + \alpha\left(KK^\theta{}_\theta - 4\pi m^2|\phi|^2\right) \quad (43)$$

- Field evolution equations

$$\dot{\phi} = \frac{\alpha}{a}\Pi + \beta\Phi \quad (44)$$

$$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \quad (45)$$

$$\dot{\Pi} = \frac{1}{(rb)^2}\left[(rb)^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right]' - \alpha a m^2\phi + 2\left[\alpha K^\theta{}_\theta - \beta\frac{(rb)'}{rb}\right]\Pi \quad (46)$$

Appendix: Boson Stars in Spherical Symmetry

- Maximal-isotropic coordinates
  - Maximal slicing condition

$$K \equiv K_i^i = 0 \qquad \dot{K}(t,r) = 0 \tag{47}$$

  - Isotropic condition

$$a = b \equiv \psi(t,r)^2 \tag{48}$$

  - They fix the lapse and shift (equivalent of fixing the coordinate system)

$$\alpha'' + \frac{2}{r\psi^2}\frac{d}{dr^2}\left(r^2\psi^2\right)\alpha' + \left[4\pi\psi^4 m^2|\phi|^2 - 8\pi|\Pi|^2 - \frac{3}{2}\left(\psi^2 K^r{}_r\right)^2\right]\alpha = 0 \tag{49}$$

$$r\left(\frac{\beta}{r}\right)' = \frac{3}{2}\alpha K^r{}_r \tag{50}$$

  - Constraint equations

$$\frac{3}{\psi^5}\frac{d}{dr^3}\left(r^2\frac{d\psi}{dr}\right) + \frac{3}{16}K^r{}_r{}^2 = -\pi\left(\frac{|\Phi|^2 + |\Pi|^2}{\psi^4} + m^2|\phi|^2\right) \tag{51}$$

$$K^r{}_r{}' + 3\frac{(r\psi^2)'}{r\psi^2}K^r{}_r = -\frac{4\pi}{\psi^2}\left(\Pi^*\Phi + \Pi\Phi^*\right) \tag{52}$$

Appendix: Boson Stars in Spherical Symmetry

- Complex-scalar field evolution equations

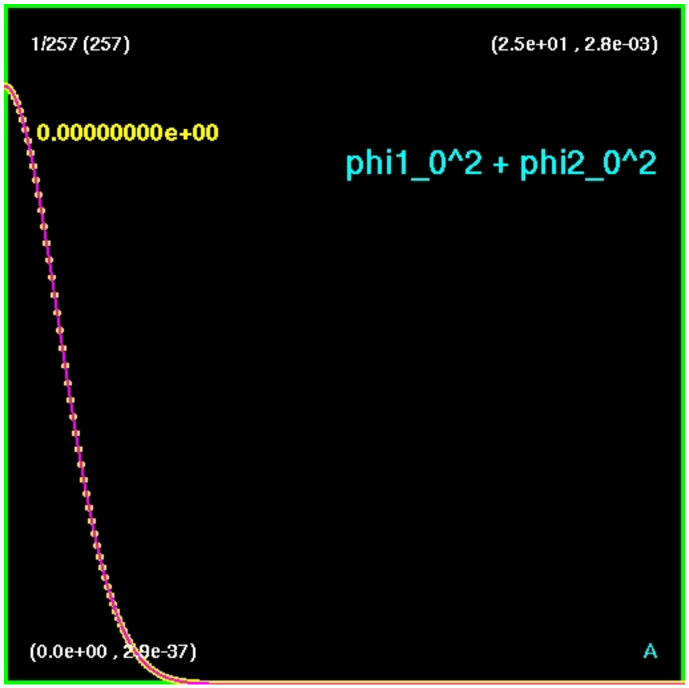
$$\dot{\phi} = \frac{\alpha}{\psi^2}\Pi + \beta\Phi \tag{53}$$

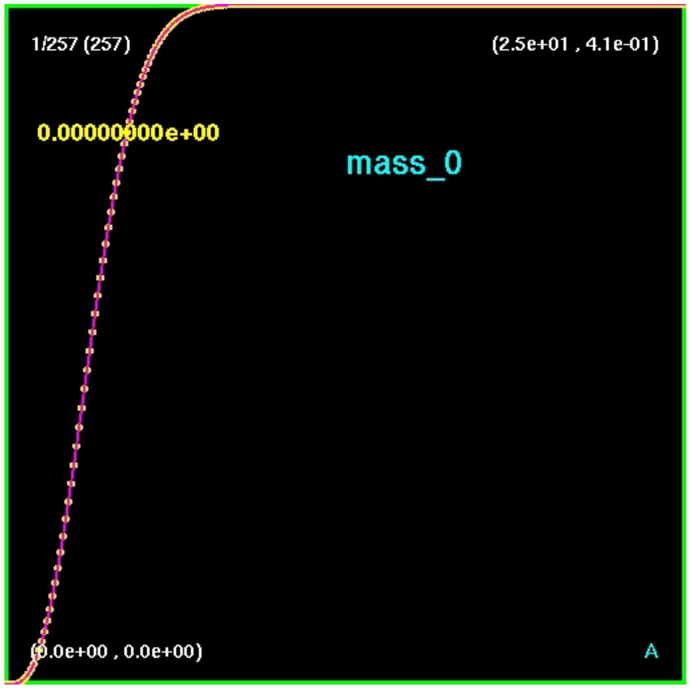
$$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{\psi^2}\Pi\right)' \tag{54}$$

$$\dot{\Pi} = \frac{3}{\psi^4}\frac{d}{dr^3}\left[r^2\psi^4\left(\beta\Pi + \frac{\alpha}{\psi^2}\Phi\right)\right] - \alpha\psi^2 m^2\phi - \left[\alpha K^r{}_r + 2\beta\frac{(r\psi^2)'}{r\psi^2}\right]\Pi \tag{55}$$

Appendix: Boson Stars in Spherical Symmetry

- These equations were coded using RNPL and tested for a gaussian pulse as initial data.





Appendix: Boson Stars in Spherical Symmetry

- Initial Value Problem

- We are interested in generating *static* solutions of the Einstein- Klein-Gordon system

- There is no regular, time-independent configuration for complex scalar fields but one can construct harmonic time-dependence that produce time-independent metric

- We adopt the following ansatz for boson stars in spherical symmetry in order to produce a static spacetime:

$$\phi(t,r) = \phi_0(r)\, e^{-i\omega t}, \qquad \beta = 0 \tag{56}$$

  where the last condition comes from the demand of a static timelike Killing vector field.

- Polar-Areal coordinates

$$K = K^r{}_r \qquad b = 1 \tag{57}$$

- Generalization of the usual Schwarzschild coordinates to *time-dependent*, spherically symmetric spacetimes. Easier to generate the initial data solution

Appendix: Boson Stars in Spherical Symmetry

- The line element

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 \,. \tag{58}$$

- The equations of motions are cast in a system of ODEs. It becomes an eigenvalue problem with eigenvalue $\omega = \omega(\phi_0(0))$

$$a' = \frac{1}{2}\left\{\frac{a}{r}\left(1 - a^2\right) + 4\pi r a \left[\phi^2 a^2 \left(m^2 + \frac{\omega^2}{\alpha^2}\right) + \Phi^2\right]\right\} \tag{59}$$

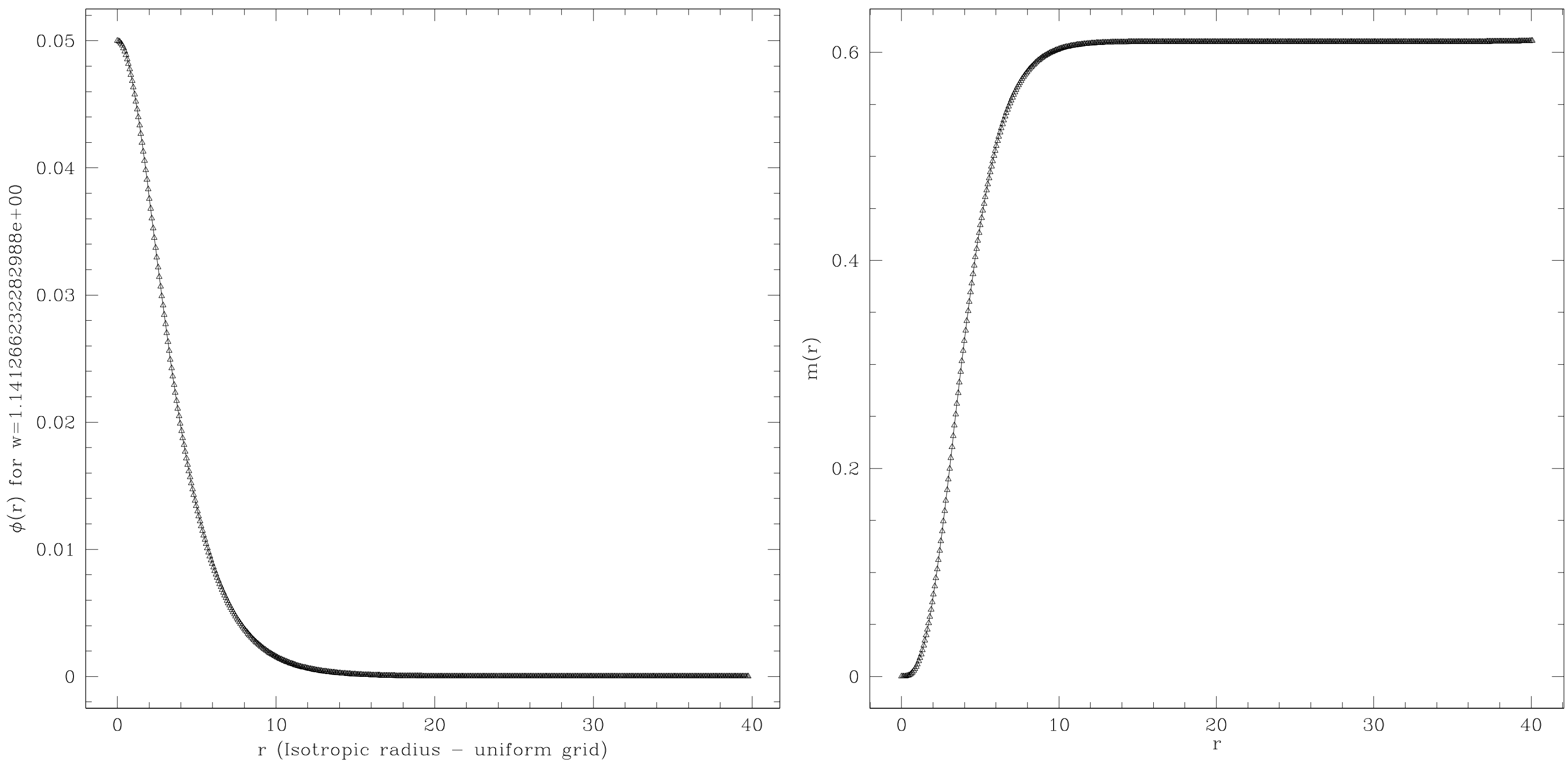
$$\alpha' = \frac{\alpha}{2}\left\{\frac{a^2 - 1}{r} + 4\pi r \left[a^2 \phi^2 \left(\frac{\omega^2}{\alpha^2} - m^2\right) + \Phi^2\right]\right\} \tag{60}$$

$$\phi' = \Phi \tag{61}$$

$$\Phi' = -\left(1 + a^2 - 4\pi r^2 a^2 m^2 \phi^2\right)\frac{\Phi}{r} - \left(\frac{\omega^2}{\alpha^2} - m^2\right)\phi a^2 \tag{62}$$

Appendix: Boson Stars in Spherical Symmetry

- Field configuration and its aspect mass function for $\phi_0(0) = 0.05$. Its eigenvalue was "shooted" to be $\omega = 1.1412862322$



- Note its exponentially decaying tail as opposed to the sharp edge ones for its fluids counterparts

Appendix: Boson Stars in Spherical Symmetry

- The ADM mass as a function of the central density and the radius of the star as a function of ADM mass. Note their similarity to the fluid stars