MAXIMUM MASS AND COLLAPSE FUNCTION FOR BOSON STAR MODELS

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Abstract. In this paper we investigate the behaviour of maximum mass for a stable boson star for several different self-interaction potentials. The results are in agreement with those in the literature for $\lambda|\phi|^4$ and $\gamma|\phi|^6$ self-interaction potentials. We also find novel features for $\lambda|\phi|^3$ theory. We started in this paper a search for a model rendering the maximum collapse function, $z = \frac{2m(r)}{r}$, (also known in the literature as coefficient of relativisticity). First, we investigated separately a series of different polynomial self-interaction potentials from $|\phi|^3$ to $|\phi|^6$. Then we started a survey of the coupling constants space. Rudimentary results for $\gamma|\phi|^3 + \lambda|\phi|^4$is shown.

1. Introduction

The initial idea of boson star evolved from the pioneering work by John Wheeler [1] on electromagnetic (EM) self-gravitating entities called geons. The idea arises from considering the gravitational attraction effects that the mass (energy) associated to an electromagnetic disturbance would be able to exert. A sufficiently energetic electromagnetic disturbance would then give rise to a gravitation attraction capable of holding the disturbance together for a long time compared to any other characteristic time of the system. This type of system was also found to require rotation to be stable. Kaup [2] picked up on the geon concept and minimally coupled a massive complex scalar field to general relativity rather than the EM field. Assuming a static spherically symmetric solution he found solutions to the coupled equations, which he called Klein-Gordon geons. These solutions were later renamed boson stars (BS).

Following the work by Kaup, Ruffini and Bonazzola [3] showed that the classical limit for the BS stress-energy tensor could be obtained by the mean value of its quantum counterpart over the ground state vector for a system of many particles. At zero temperature, all the bosons in the system will occupy this ground state, forming a Bose-Einstein condesate (BEC). Then a BS is a self-gravitating compact object (compact, in the sense that its radius is of the order of Schwarzschild radius) composed of a large number of scalar particles in their ground state (BEC), and described classically by a complex scalar field minimally coupled to gravity.

To date there exists no known fundamental scalar particle. Scalar fields, however, hardly represent a revolution in cosmology as their existence has been studied for quite some time. Examples are the inflation field, proposed by Guth in 1981 [6] and the dilaton field which is the fundamental field in bosonic string theory. There is of course a demand for the existence of the massive Higgs boson, which is currently being sought by the particle physicists [4]. Scalar particles have been proposed as a good candidate for, or at least as a component making up a good fraction of, the dark matter in the universe.

Certainly, the study of the collapse of such a boson cloud of scalar particles into a boson star could lead to a better understanding of astrophysical phenomena. The field of gravitational lensing has achieved quite some maturity in recent years and could be helpful in its detection as well as in the determination of its properties. Boson stars should exhibit distinct lensing effects, some of which have already been determined. Finally, as another example of astrophysical speculation, since boson stars could achieve a very large size, they could offer an alternative to super black holes in galactic centers.

Despite all their possible astrophysical/particle physics applications, studies on boson stars are strongly motivated by the simplification that this matter model introduces in the system of equations when compared with their fermionic counterparts. The dynamics of the scalar field is governed by a partial differential equation (PDE), viz, the Klein-Gordon equation, that does not develop any kind of singularity from a smooth initial data. Therefore, there will not be any problems with shocks, low density regions, ultrarelativistic flows, etc in the evolution of this kind of matter as opposed to fluids (fermionic matter). Therefore, the scalar field becomes a tempting model to investigate the strong-field dynamics of gravitationally compact objects and to give us some insight about the dynamics of its fermionic counterpart.

Some features of both the fermionic and bosonic system can easily be noticed from the start. For example, in spherical symmetry we can parameterize the family of solutions by the modulus of the field at $r = 0$, the central field, ϕ_0, which is analogous to the central density for perfect fluid stars, governed by the TOV equations and the ideal equation of state $P = K\rho_o^{\Gamma}$. A noticeable difference between the models though, shows up in the configuration of a boson star in equilibrium. Its tail expands, in principle to infinity, unlike the tail on fluid model stars which have sharp edges.

The gravitational equilibrium of such gravitational solitons has already been investigated but still raises lots of interest. Boson stars are prevented from collapsing gravitationally by the pressure that stems from the Heisenberg uncertainty principle. Like their fermion counterparts, neutron stars and white dwarves, boson stars also have a limiting ADM mass below which the star is stable against complete gravitational collapse into a black hole (BH). As for the neutron star case (where the degeneracy pressure provided by the Pauli exclusion principle provides the repulsive force), we can derive an expression for the maximum possible mass. This turns out to be $\sim M_{pl}^3/m^2$, where M_{pl} is the planck mass and m the scalar field mass, while the maximum mass of a non-self-interacting boson star is $\sim M_{pl}^2/m$. This comes from the fact that we can claim the boson particles within the star are confined to a region R, and thus via the uncertainty principle we have $p \cdot R \sim \hbar$. For moderately relativistic boson stars, $p = mc$, and so we get $R \sim \hbar/mc$. Equating this to the Schwarzschild condition, we have;

$$R \sim \frac{\hbar}{mc} = \frac{2M_{max}G}{c^2} \rightarrow M_{max} \sim \frac{M_{Pl}^2}{m} \tag{1}$$

The simplest variation of the standard boson star model generally consists of adding self-interaction terms to the usual massive Klein-Gordon Lagrangian, such as $\lambda|\phi^4|$, studied by Colpi et al. [5] and several others. As mentioned before, stable stars are formed when there is a balance between all the forces acting on its constituent matter. In this case it is a balance between the gravitational force, Heisenberg's uncertainty principle, and the attractive/repulsive self interaction between constituent particles. So the size and mass of the star greatly depends on the mass of the individual bosons, and on the effect of self interaction terms. In principle, then boson stars can exist in a very wide size range, from microscopic to cosmologically significant scales.

2. Overview

The aim of this project was to investigate boson stars with different effective potentials arising from various models. The motivation was twofold; first, we decided to extend the results for boson star maximal mass obtained by Colpi et al. [5] to several other potentials. The list of potentials to work with were initially based on the possible physical relevance that a particular potential could have in the inflation scenario. Following the work by Schunck-Torres [7], we implemented the cosh-Gordon, sine-Gordon and Liouville boson stars in order to look for any interesting behaviour of the maximal BS mass. As we had problems in the implementation for all but the sine-Gordon, we then decided to move on for polynomials potentials in $|\phi|$ since any exotic potential can ultimately be expanded in a polynomial series.

We also originally intended to include the dilaton, but scrapped it because being a real field, it would not satisfy our ansatz. Although this would be an interesting potential to investigate, it was agreed that it would be best left for another paper, and would likely follow the work performed by Gradwohl and Kaelbermann [11]. Basically several of our potentials are also valid for the real scalar field, but we choose to investigate the more general form of a scalar field, since it has a stronger likelihood of physical realization, because only complex scalar fields can carry charge.

We look at the basic massive Klein-Gordon model, the massive Klein-Gordon model with $\gamma\phi^3$ self interaction, the massive Klein-Gordon model with $\lambda\phi^4$ self interaction, and the massive Klein-Gordon model with $\eta\phi^6$ self interaction.

We also investigate the massive Klein-Gordon model with $\gamma\phi^3 + \lambda\phi^4$ self interaction. However these results are only preliminary. A full parameter survey was not performed due to time constraints.

A desire to get new and exciting behaviour for the usual and general potentials lead us to investigate the stability against collapse problem for those polynomial potentials.

For a fluid star, the Schwarzschild limit is defined as the minimum coordinate radius that a mass can have under static equilibrium. This is a well know result that comes from the search for possible interior fluid sources for an external Schwarzschild spacetime solution. In order to study the stability of a star against gravitational collapse, a useful parameter, the collapse function (also known in the literature as coefficient of relativisticity [8]) is defined as the ratio of the Schwarzschild radius to the radius determined by the mass configuration of the star, $z = 2m(r)/r$. Note that $z = 1$ defines the event horizon. Then a necessary condition for a static solution is that $z \leq 1$. From the condition which states that the static Killing vector is time-like everywhere, a second condition for a fluid static solution is imposed on z: $z \leq 8/9$. As an example of the consequence of this statement, when z reaches 8/9, for a star with uniform density, the central pressure of the star would diverge to infinity.

The analogue for boson stars has not yet been established (except for the obvious limit $z \leq 1$). The main property investigated in this report, besides the maximum mass for the systems, is the maximum $z = \frac{2m(r)}{r}$ for each of the potentials. This was then expected to produce, as an ultimate goal, a map of the the maximum z as a function of the coupling coefficients for a particular self-interaction potential. This value would correspond to the most compact static and stable boson star of all the potentials and most of its couplings investigated so far.

We first look at the reason we expect a maximum mass for a given system. To begin with the mass we are interested in calculating is the ADM (Arnowitt, Deser, Misner) mass. The easiest approach to this is the use of the mass aspect function, which comes from one of the metric components;

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2}$$

where m(r) above is the mass that the areal radius encapsulates. This metric is used because we are expecting a spherically symmetric solution. Since we demand asymptotic convergence with the Schwarzschild metric for our system, we compare the above metric with;

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dr dt + a^2 dr^2 + R^2 d\Omega^2 \tag{3}$$

However with the static assumption, we have that $\beta = 0$. We further note that the choice of polar slicing $K = K_r^r$ and areal coordinates, $b = 1$, reduce the problem of solving the Einstein-Klein-Gordon system to that of finding one free parameter, the eigenvalue for the system.

The mass aspect function m(r) comes from the matching of the Schwarzschild metric to the spherically symmetric metric as r $\to \infty$;

$$\left(1 - \frac{2m(r)}{r}\right)^{-1} = a(r)^2 \tag{4}$$

which gives us our mass aspect function;

$$m(r) = \frac{r}{2}\left(1 - \frac{1}{a(r)^2}\right). \tag{5}$$

In all stellar models we expect a continuum of solutions for any given central density. However the most physically relevant stars will occur along the stable branch, that is the stellar models that exist up to the maximum value on the mass aspect function plotted against the central density. If we look at figure 2, we see the maximum mass exists at central density 0.08, so that central densities less than this value will result in stable stars, central densities larger than this value will result in unstable, oscillating stars, which are beyond the focus of this paper. In many papers on this topic, this is the value they were interested in calculating. Thus, several relationships were forged on this idea, especially in the paper by Colpi et al., where an asymptotic relationship between the maximum mass as a function of the self-interacting coefficient is established. In this paper we re-enforce this relationship.

Based on a suggestion by Matt Choptuik, we needed to further investigate another scalar property of the star, that is $z = \frac{2m(r)}{r}$. This quantity is simpler to calculate than the mass aspect function;

$$\left(1-\frac{2m(r)}{r}\right)^{-1} = (1-z)^{-1} = a(r)^2 \tag{6}$$

so we are left with;

$$z = 1 - a(r)^{-2} \tag{7}$$

The idea behind this value for a given star model is that z, tells us the gravitational effect of the star. z is expected to take on values between 0 and 1. As z approaches zero, the gravitational effects are minimized. However at 1, we have a coordinate singularity. (see equation 2) This tells us that the star is the size of the event horizon, which in principle tells us that the star with this value of z, would be the largest possible star with the given set of parameters.

3. Formalism

First we must explain the basic equations of our system. We start with the Lagrangian scalar with a general potential;

$$L_\phi = \frac{1}{2}\left(\nabla^\mu\phi\nabla_\mu\phi^* + U(|\phi|^2)\right) \tag{8}$$

with the stress energy tensor

$$T_{\mu\nu} = \frac{1}{2}\left[(\nabla_\mu\phi\nabla_\nu\phi^* + \nabla_\nu\phi\nabla_\mu\phi^*) - g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi + U(|\phi|^2)\right)\right] \tag{9}$$

which taking the variation of the Lagrangian with respect to the field leads us to the Klein-Gordon equation with a general potential;

$$\nabla^\mu\nabla_\mu\phi = \frac{dU(|\phi|^2)}{d|\phi|^2}\phi \tag{10}$$

It is also important to note that we are using spherical symmetry, and a static spacetime. With this we adopt an ansatz for our solution;

$$\phi(r,t) = \phi(r)e^{-i\omega t} \tag{11}$$

So with this information we get our equations of motion with general potentials;

$$a' = \frac{1}{2}\left\{\frac{a}{r}(1-a^2) + 4\pi a r\left[a^2U(\phi_0^2) + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right]\right\} \tag{12}$$

$$\alpha' = \frac{\alpha}{2}\left\{\frac{1}{r}(a^2-1) + 4\pi a r\left[\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 - a^2U(\phi_0^2) + \Phi_0^2\right]\right\} \tag{13}$$

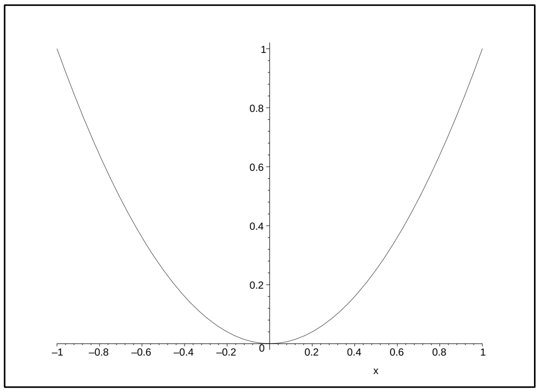

Figure 1. The massive potential

$$\phi_0' = \Phi_0 \tag{14}$$

$$\Phi_0' = \left(\frac{dU(\phi_0^2)}{d\phi_0^2} - \frac{\omega^2}{\alpha^2}\right) a^2\phi_0 - \left(1 + a^2 - 4\pi r^2 a^2 U(\phi_0^2)\right) \frac{\Phi_0}{r} \tag{15}$$

With the equations of motion established we are now free to investigate the different potentials of interest. Note that a full derivation of the 3+1 equations of motion may be found in the appendix.

4. Massive Boson

With the underlying physics explained we first look at the most basic assumption, the massive, self interacting boson star. The potential for this case is simply;

$$U\left(|\phi|\right) = m^2|\phi|^2 \tag{16}$$

For simplicity we can scale our problem so that mass is no longer considered explicitly. Basically this has the same effect as setting the mass to $m = 1$.

This potential has been investigated in too many papers and thesis to list, however the most recent that comes to mind in [10]. It produces fairly standard results which we have reproduced, and may be seen in figure 2.

In figure 2 we obtain a maximum mass for the stable branch of the boson star of 0.633 M_{Ch}, which corresponds to $\phi_o = 0.08$. We, therefore, have an analogue of the Chandresehkar mass limit, above which no static configuration exists. This maximum mass is a relativistic effect, if this were a Newtonian theory, there would be no such upper bound for the boson star. For this

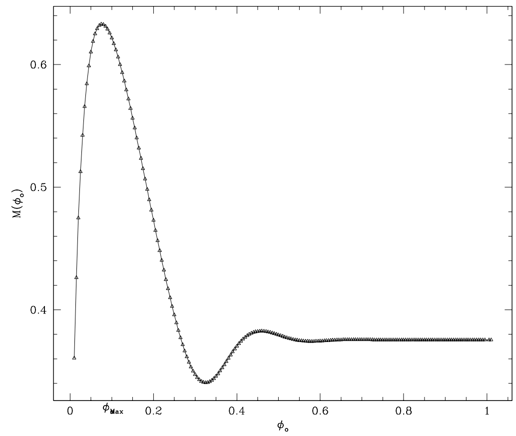

FIGURE 2. The total mass of a stable boson star as a function of the field at the centre of the star. Here we clearly label what we mean by ϕ_{max}

potential we find that the stable solutions exist for $\phi_o < 0.08$ and the rest are unstable.

We see in the magnified section of the mass function in figure 3, that the apparent plateau on the far right of the graph in figure 2, is actually comprised of rapidly damping oscillations, which are of a very small amplitude, which makes them difficult to see in most papers.

We have also included several fairly standard plots for the massive boson star as a function of the areal radius r. Ultimately we see exactly what is expected for all values of the central density, mass, lapse, and the collapse function. See figure 4.

Furthermore, we look at the eigenfrequency as a function of central density, no rescaling was performed. These plots are produced in figure 5.

There is one drawback to this potential, that it leaves us with the mass of the individual boson particles on the order of the Planck mass. Since that would be unphysical, this theory must be rejected as the true form for the massive Boson star. The maximum z obtained for this particular theory is 0.22.

To determine the maximum value of z, for every value of the coupling coefficient, one has to run two sets of data. First of all, one must find the maximum values of z for every ϕ_o, of which there will be one per value of central density observed up to the maximum ϕ_o as described earlier. Then with these values, one must select the maximum value in this data set. This value is considered to be the maximum z for that particular value of the coupling coefficient. As

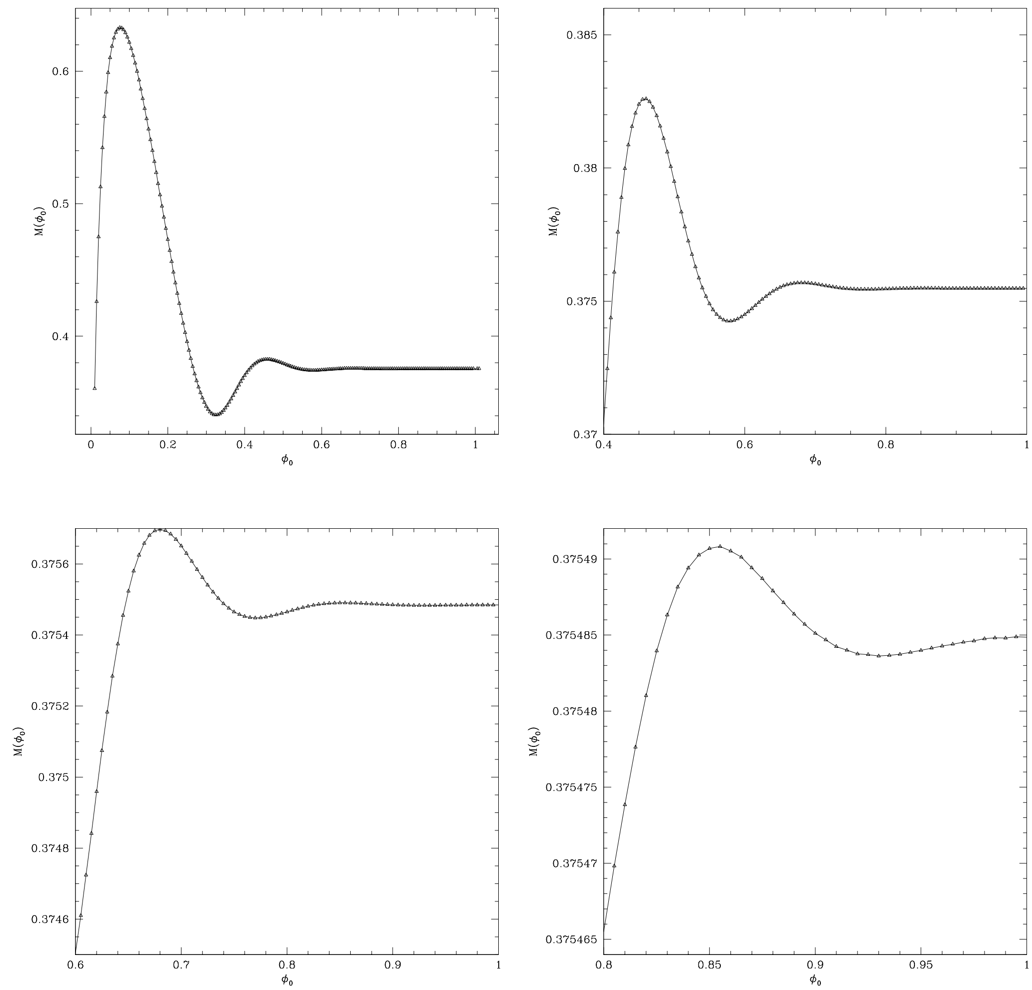

FIGURE 3. The total mass of a stable boson star as a function of the field at the centre of the star. The plots all show the same data, and clearly indicate the self-similarity of the relation.

an observation we noted that this value of the maximum z, always coincided with the maximum ϕ_o as associated with the maximum mass.

In the massive case with no self interaction terms, there is only one coupling coefficient, the mass, which was scaled to unity.

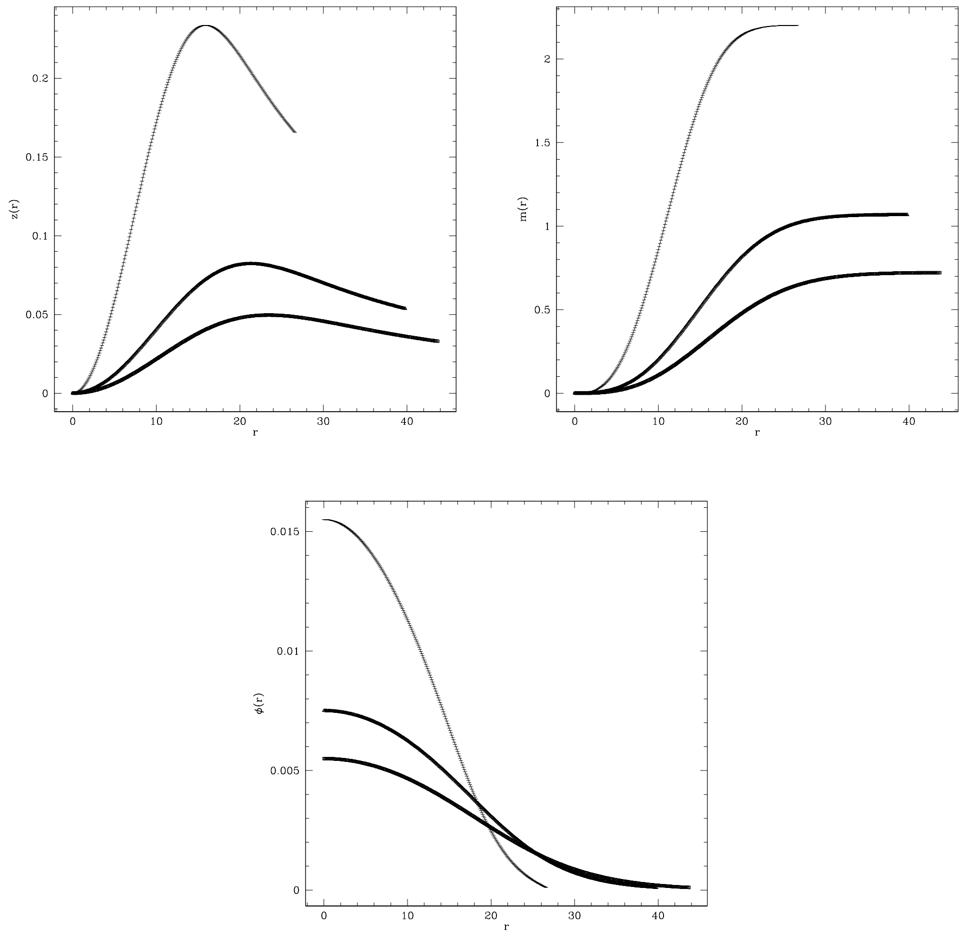

FIGURE 4. The collapse function, z; mass aspect function, $m(r)$; and the scalar field, ϕ, as a function of areal coordinate r. In all cases and the central field values are $\phi_0 = 0.0155$, $\phi_0 = 0.0075$ and $\phi_0 = 0.0055$ from top to bottom. The m(r) plot clearly displays the asymptotic nature of the mass as a function of the areal coordinate. The z(r) plots also show that there is a clear maximum z for each value of ϕ investigated. Finally in the ϕ(r) plot we see the asymptotic tail which is characteristic of the boson star, a feature which does not exist for the fermionic star.

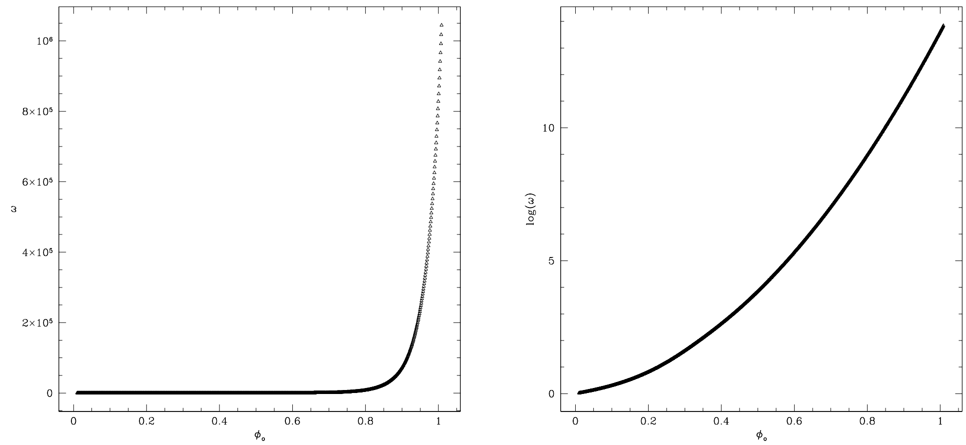

FIGURE 5. The angular frequency as a function of the central density.

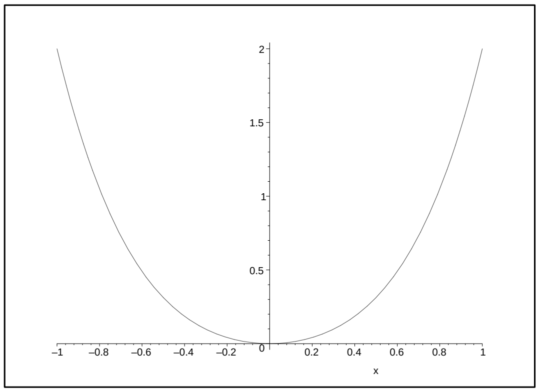

FIGURE 6. The massive potential with $\Lambda\phi^4$ self-interaction

5. $\lambda\phi^4$ SELF-INTERACTION

Shortly after the massive boson theory was proposed – and the basic flaw pointed out – Colpi et el. reformulated the theory, modifying the potential to include self interactions. The standard model used for the self interacting scalar bosons is the $\lambda\phi^4$ model. This model was likely selected since it is well accepted among field theorists as a toy model. The physics behind it is well understood, and it has a well defined ground state.

We investigated this model using our scaled equations, so we exchanged λ for a new coefficient, $\Lambda = \lambda/m^2$, which we again implemented by setting $m = 1$ for simplicity.

We should note that we chose a different definition for the coefficient λ than used in the Colpi paper. Since the conversion factor can be absorbed into the coefficients, we will disregard this in our analysis, treating λ the same as others have. It is, however, important to note that a direct comparison between our values and those from the original paper will require a conversion factor of 4π.

To solve this system we use the potential

$$U(|\phi|^2) = |\phi|^2 + \Lambda|\phi|^4$$

that can be seen in figure 6. In figure 7 we have a sample of the different Λ's investigated.

We encountered significant difficulty in obtaining ground state solutions to the Klein-Gordon equation for values larger than 1600. The shooting method used would have required such a small step size, that pursuing any higher values of Λ would have been too computationally expensive.

With the data at hand we do find the interesting result that the z versus Λ plot, seen in figure 10 shows asymptotic behaviour for large lambda, with a maximum value for the gravitational effect for a star with this potential at

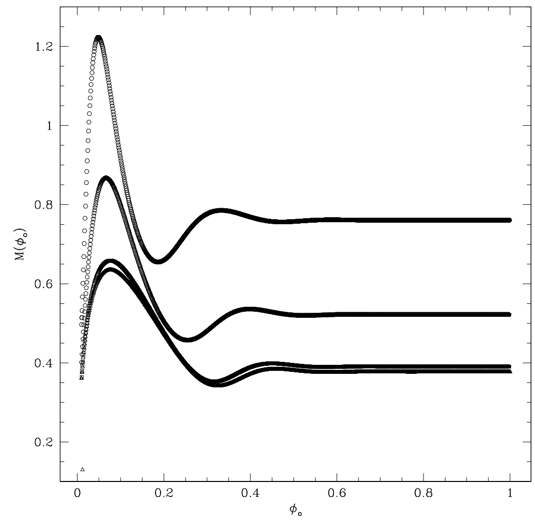

FIGURE 7. The ADM mass as a function of central density for the $\lambda|\phi|^4$ potential. From bottom to top, we have $\Lambda = 0$, 1, 10, 100

around $z = 0.34$. So, if we are interested in creating a highly relativistic boson star, we can hardly be satisfied with this particular theory.

As discussed by Colpi et al., a boson star in the ϕ^4 self-interacting model can, unlike in the ϕ^3, have a physically realistic boson mass. Our goal, however, is not merely to find a sensible model, but also to search for one that has, as mentioned, as relativistic a star as possible.

Earlier we had established a relationship between the maximum mass of the massive boson star with no self-interactions in equation 1. One can further establish a relation between the maximum mass of a boson star with self interaction terms. For that, we must consider the relative strength of the interaction terms. The only condition where the self-interaction terms are really effective is when they are on the order of the mass term, ie when;

$$\frac{V(|\phi|)}{m^2|\phi|^2} \sim O(1) \tag{17}$$

where $V(|\phi|)$ contains all the self-interaction terms. As is worked out in [9] with the ϕ^4 interaction, we have;

$$\frac{\lambda|\phi|^4}{m^2|\phi|^2} \sim O(1) \tag{18}$$

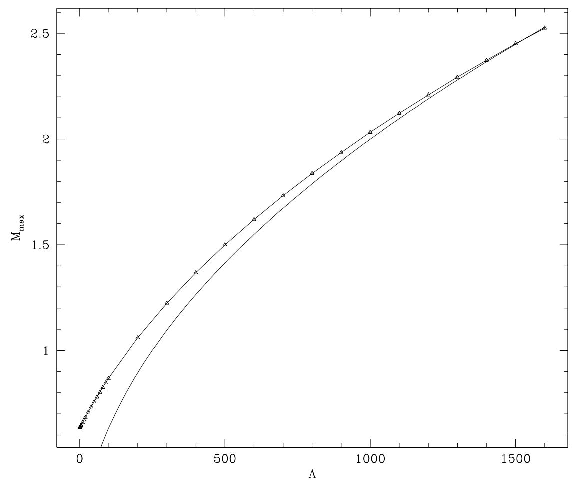

FIGURE 8. Here we have the maximum mass of the boson star as a function of the coupling coefficient. The solid line indicates the asymptotic value of this theory at $\Lambda^{1/2}M_{ch}$ The amplitude of this asymptotic relation is 0.063, in exact agreement with the Colpi results. It is also important to note that unlike the Colpi paper, no data smoothing was applied.

so

$$\frac{\Lambda|\phi|^2}{M_{Pl}^2} \sim O(1) \tag{19}$$

where we have made the substitution $\Lambda = \lambda M_{Pl}^2/m^2$ and the $|\phi| \sim m$ instead of $|\phi| \sim M_{Pl}$ as was the case in the no self-interaction potential. With this we have that the radius is rescaled to be $R_\lambda \sim \Lambda^{1/2}/m$ and the density $\rho = m^2|\phi|^2 + \lambda|\phi|^4$ and so the maximum mass becomes;

$$M_{max} = \rho R^3 \sim \Lambda^{1/2}M_{Pl}^2/m \sim \lambda^{1/2}M_{Ch} \tag{20}$$

In figures 9 and 10, we look at both positive and negative values for the coupling coefficient. The positive coupling coefficients allow the ϕ^4 theory to be bounded, which may be seen in figure 6. However for the negative coupling coefficient the potential is unbounded, which may be visualized by inverting the function in figure 6. As expected for the unbound theory, the value of z goes to zero, as the coefficient becomes very negative. This was interpreted as the mass of each particle going to a value much less than the Planck mass,

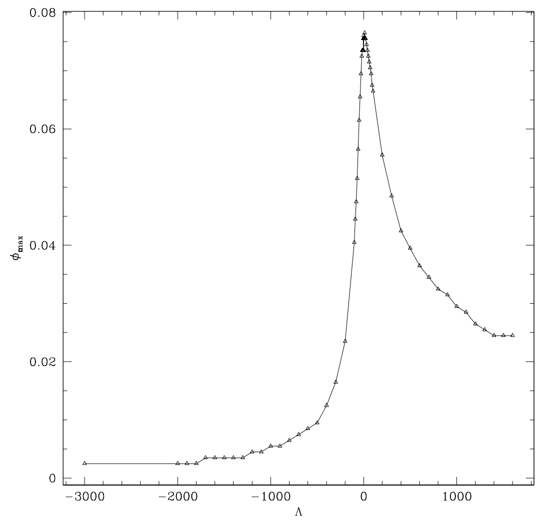

FIGURE 9. Here we have the value of the ϕ_{max} at the location of the maximum z for a given theory. It is important to note that the location of the central density decreases as a function of Λ which hinders our ability to assess the value of maximum z, for large Λ, since for very large Λ the location of ϕ_{max} will be below our resolution of ϕ_o. The small cluster of points forming the plateau on the curve is attributed to the resolution of our step sizes in ϕ in solving the equations of motion.

since this would be an attractive theory between constituent particles and thus the mass would be smaller than the Planck mass.

Since this process produces a particle which has an unrealistic mass, it must also be discarded as a legitimate theory.

6. $\gamma\phi^3$ SELF INTERACTION

Unlike ϕ^4 theory, this potential, shown in figure 11, has the drawback of not being physically meaningful on its own. In the ultraviolet regime, the ϕ^3 theory is not renormalizable without being coupled to a higher order interaction such as ϕ^4.

On its own, it is still of great interest to field theorists as it also proves to be a useful model for understanding quantum field theory, ignoring the above mentioned caveat. In the case of boson stars, no one has yet done analysis of potentials involving a complex ϕ^3 term. This is treated in this paper without any coupling, specifically for gaining insight into the system.

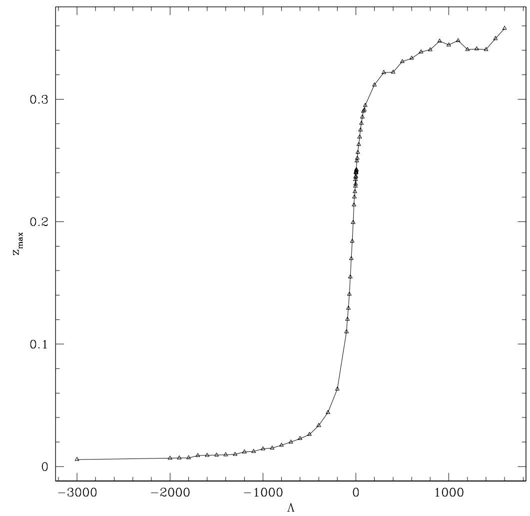

FIGURE 10. Here we see the maximum z as a function of Λ for the ϕ^4 theory. We see something like an asymptotic relation for large Λ.

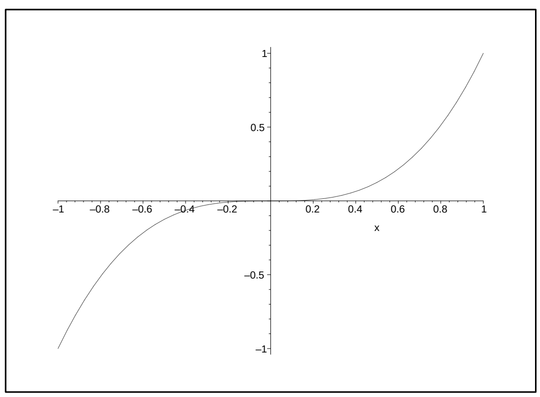

FIGURE 11. The massive potential with $\Gamma\phi^3$ self-interaction

The form of the potential we used for this theory is;

$$U(|\phi|^2) = |\phi|^2 + \Gamma|\phi|^3 \tag{21}$$

where, similar to the previous treatment, the potential coefficient, Γ, is rescaled to absorb constants and the mass term.

Consistent with the analysis of the ϕ^4 model, we investigate both the mass aspect function, and the maximum z as functions of the free parameter, ϕ_0. Again, we are looking for the highest possible value of the collapse function, z, to get as relativistic a star as possible.

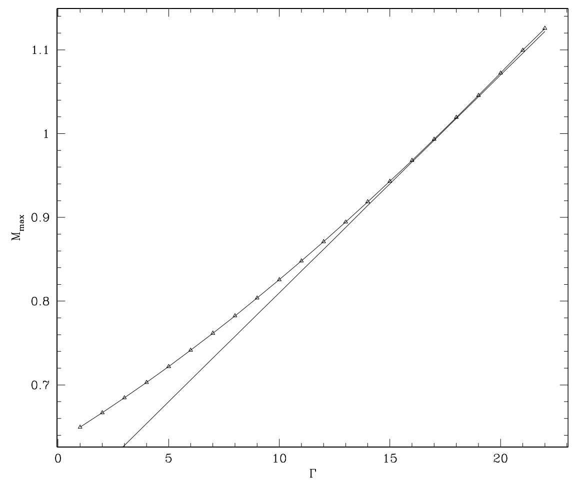

FIGURE 12. The maximum mass as a function of the coupling coefficient Γ. We do observe the linear asymptotic relationship speculated for the ϕ^3 theory. The slope of this relation is 0.025

We see in figure 14 that with a positive coupling constant, the value for z is still smaller than in the ϕ^4 theory and we need Γ to be negative to equal the final value obtained in ϕ^4 theory.

We have seen in equation 20 the relationship between the maximum mass and the coupling constant. Based on the work done in [9], we speculate a relationship between the maximum mass and the coupling constant for ϕ^3 theory. The relationship appears to be of the form;

$$M_{max} = \rho R^3 \sim C_n^{\frac{1}{n-2}} M_{Pl}^2/m \tag{22}$$

when we are dealing with ϕ^n theory, and C_n is the coupling coefficient for that theory. So when we are looking at ϕ^3 theory we expect an asymptotic relationship;

$$M_{max} = \rho R^3 \sim \Gamma M_{Pl}^2/m \tag{23}$$

Thus for a ϕ^3 potential, we expect an asymptotic linear relationship between the maximum ADM mass, and the coupling coefficient Γ.

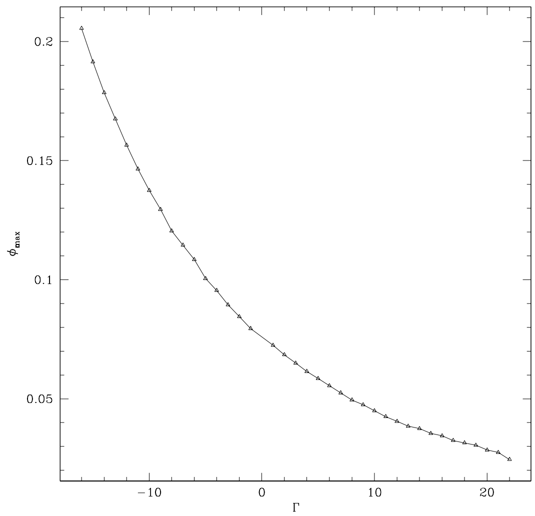

FIGURE 13. The value of ϕ_{max} as a function of Γ. For the positive coupling coefficient we note that ϕ_{max} decreases as Γ increases, tending to zero very quickly. However unlike the ϕ^4 case, when we consider the negative coupling constant ϕ_{max} increase as $|\Gamma|$ increases.

7. $\xi\phi^6$ SELF INTERACTION

Another possible self interacting theory is the $\xi\phi^6$ potential. Since this theory has a coefficient with a negative mass dimension, it is non-renormalizable. However, unlike the odd powered potentials, this one, which can be seen in figure 15, has a well defined ground state, and thus if we avoid the ultraviolet divergences, this theory stands a chance of being considered physically meaningful.

The form of the potential used in this theory is;

$$U(|\phi_o|^2) = |\phi|^2 + \xi|\phi|^6 \tag{24}$$

where again the potential is rescaled to absorb all constants and mass terms.

In good consistent form we investigate both the Mass aspect function, and the maximum z as functions of the free parameter. Again we are looking for a value of z that would allow for the star to be as large as physically possible.

We see in figure 18 that with a positive coupling constant, the value for z is still not large enough to achieve this goal. However it is interesting to note that on its own, this theory reaches values of z, which are larger than that of ϕ^4 theory. Again turning to the results in [9], we expect a maximum mass

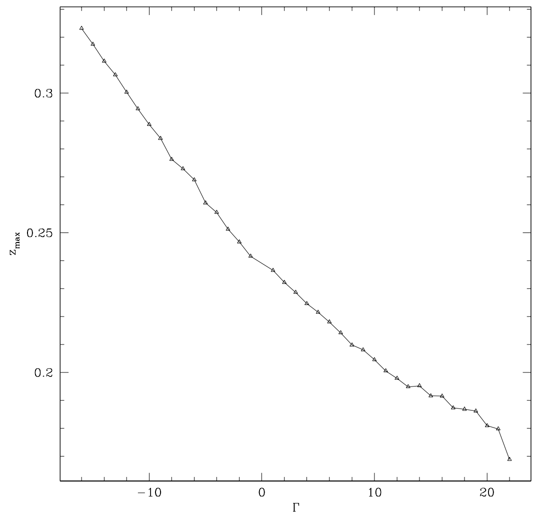

FIGURE 14. The maximum z as a function of Γ for the ϕ^3 theory. We see a rapid decrease of the maximum z as Γ increases. For the value of maximum z with a negative coupling constant we see a linear increase, with no clear indication of a maximum value.

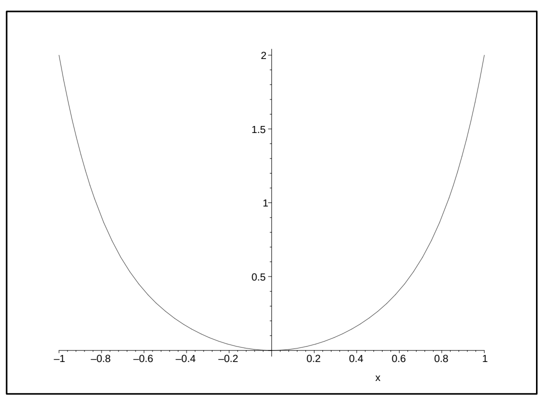

FIGURE 15. The massive potential with $\eta\phi^6$ self-interaction

relation;

$$M_{max} = \rho R^3 \sim \xi^{1/4} M_{Pl}^3/m^2 \tag{25}$$

From figure 16, we see that, as mentioned in the case of both ϕ^3, and ϕ^4 theory, shown in figures 8 and 12, this asymptotic relation holds.

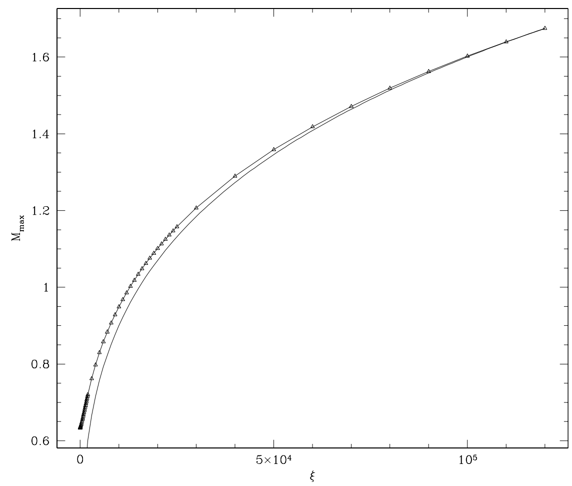

FIGURE 16. The maximum ADM mass as a function of the coupling coefficient ξ. Here we have an asymptotic relation for this theory as $\Gamma^{1/4}$. The amplitude for this asymptotic relationship was found to be 0.09. This value disagrees with that found in [9], however this is not unexpected since they admit to having significant numerical errors in their obtained values.

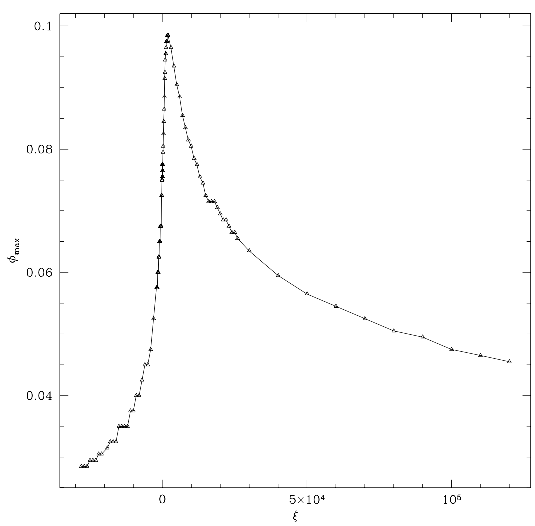

FIGURE 17. The value of the maximum ϕ, as a function of ξ. We see that the value of ϕ_{max} decreases as $|\xi|$ increase.

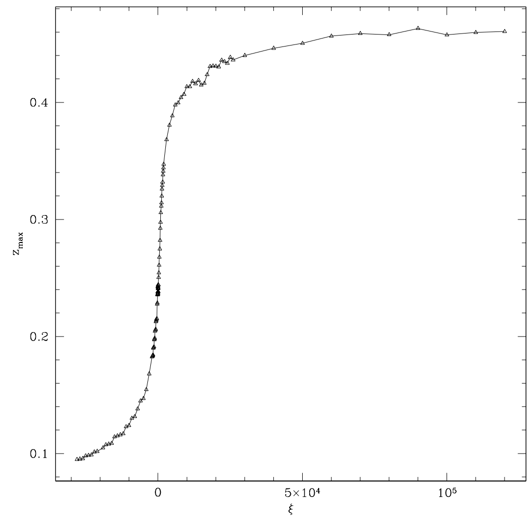

FIGURE 18. The maximum z as a function of ξ. We see similar behaviour to that of the ϕ^4 theory. However in this case we were able to observe a asymptotic limit for the maximum $z \approx 0.46$. This indicates that one may expect a similar limit for the ϕ^4 theory.

8. 2 PARAMETER SURVEY

With the behaviour of the individual ϕ^n theories established, we now turn to the more interesting problem of determining the behaviour of mixing these theories. We do a small parameter survey of the $U(|\phi|) = \Gamma|\phi|^3 + \Lambda|\phi|^4$ potential. This potential, shown in figure 19 has the form;

The resulting relationship between the maximum z, and the two coupling coefficients are shown in figure 20. Unfortunately, due to time constraints, a proper 2 dimensional mapping was not completed, and only preliminary results were obtained.

In this plot we see that the mixing of these two potentials does not increase the overall value of the maximum z. However from the investigation of the individual theories one would not necessarily expect a large increase in maximum z, with the positive coupling coefficients on the ϕ^3 term. It might have been a little more fruitful to investigate the mixture of these two theories with negative coupling coefficient on the ϕ^3 term. However that will be left for a later investigation.

We do see the general trend that for large values of Γ and small values of Λ we see a decrease in the value of maximum z. However for large values of Λ we find that the system is no longer significantly affected by the Γ contribution.

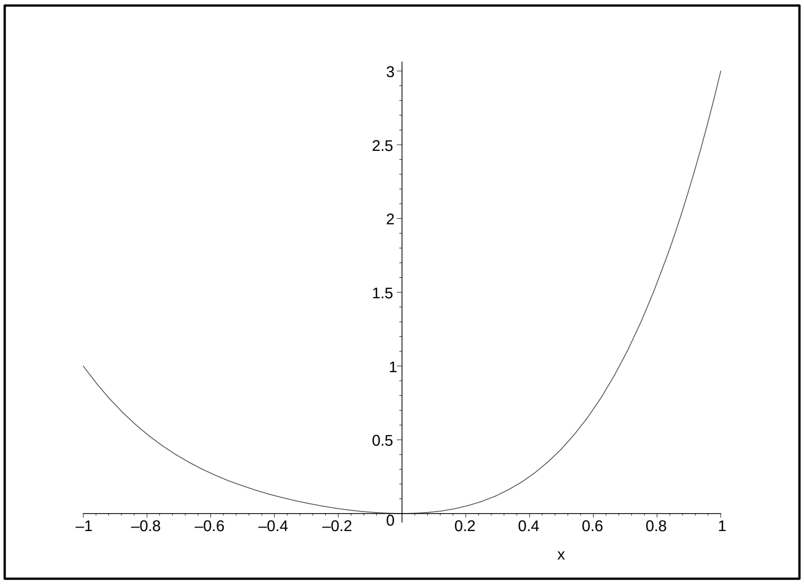

FIGURE 19. The maximum z as a function of the coefficients Γ and Λ, from the $\Gamma|\phi|^3 + \Lambda|\phi|^4$ potential.

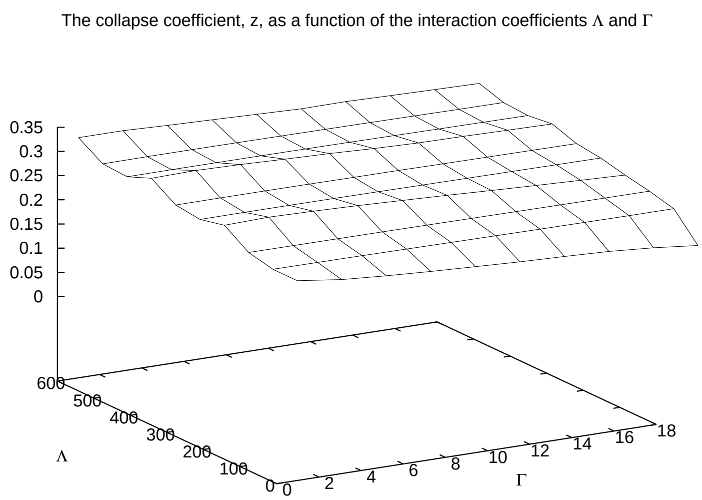

FIGURE 20. The maximum z as a function of the coefficients Γ and Λ, from the $\Gamma|\phi|^3 + \Lambda|\phi|^4$ potential.

We also note oscillatory behaviour seen for small Γ. To determine if this is a true feature of the system, we will have to investigate this system to much higher resolution.

9. Future Work and Conclusions

The conclusion of this report is that the investigation into the maximum value of the collapse function revealed different values for different theories. It would appear that an asymptotic limit exists for both the ϕ^4 and ϕ^6 theories. Our data, however, does not indicate that there is any limit to the collapse function for the negative ϕ^3 potential. It is, therefore, easy to speculate that the $-|\Gamma||\phi|^3 + |\Lambda||\phi|^4$ potential may produce a boson star with a maximum z that approaches unity.

Broader applications of this research may extend to charged scalar fields, where one must take into consideration the Maxwell interactions between the scalar particles, so an extra repulsion term will be incorporated, minimizing the required coupling of the self-interaction theories. Of course, one could spend a lifetime re-investigating all the known theoretical potentials for scalar stars, using this new approach to analysis.

As mentioned earlier a proper 2 parameter survey was not completed, and for future work we would like to endeavor to finish this task, by surveying the parameter space of positive Λ and both the positive and negative spaces for Γ to verify our speculation.

This research could also be used to map out full relationships in the generalized $U(|\phi|) = \sum_{i=2}^{\infty} C_i |\phi|^i$ theory. Which would be expected to encapsulate all possible potentials. Including the exotic cosh-Gordon, and sin-Gordon potentials, as all analytic functions may be approximated to any accuracy using Taylor series expansions.

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Appendix A. Derivations

We begin with an Einstein-Klein-Gordon system with a self-interaction potential

$$\mathcal{L}_\phi = 1\frac{1}{2}\left(\nabla^\mu\phi\nabla_\mu\phi^* + U(|\phi|^2\right) \tag{26}$$

and the stress energy tensor

$$T_{\mu\nu} = \frac{1}{2}\left[(\nabla_\mu\phi\nabla_\nu\phi^* + \nabla_\nu\phi\nabla_\mu\phi^*) - g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi + U(|\phi|^2\right)\right] \tag{27}$$

We vary the Lagrangian with respect to the field to obtain the equation of motion.

$$\nabla^\mu\nabla_\mu\phi = \frac{dU(|\phi|^2)}{d|\phi|^2}\phi \tag{28}$$

Gauss-Codazzi equations (constraints to be satisfied at each slice)
(29)

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho \qquad \rho = T_{\mu\nu}n^\mu n^\nu \qquad D_jK^{ij} - D^iK = 8\pi j^a \qquad j^\mu = -\perp(T^{\mu\nu}n_\nu)$$

Line element in 3+1 form

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j w/t^\mu = \alpha n^\nu + \beta^\mu \tag{30}$$

Evolution equations

$$\begin{aligned}
\mathcal{L}_t\gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\mathcal{L}K_{ij} &= -D_iD_j\alpha + \alpha\left\{^{(3)}R_{ij} - 2K_{ik}K^k_j - 8\pi[s_{ij} + \frac{1}{2}\gamma(s-\rho)]\right\} \\
&\quad + \beta^kD_kK_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k
\end{aligned}$$

where $S_{\mu\nu} \equiv \perp T_{\mu\nu}$ and $S = \gamma^{ij}S_{ij}$
Spherically symmetric spacetime

$$ds^2 = (-\alpha^2 + \alpha^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2dr^2 + r^2b^2d\Omega^2 \tag{31}$$

$$K^i_j = \mathrm{diag}(K^r_r, K^\theta_\theta, K^\theta_\theta) \tag{32}$$

Non-vanishing components of the matter source terms: auxilliary filds

$$\Phi \equiv \phi' \qquad \Pi \equiv \frac{a}{\alpha}(\dot{\phi} - \beta\phi') \tag{33}$$

$$\begin{aligned}
\rho &= \frac{|\Phi|^2 + |\Pi|^2}{2a^2} + \frac{U(|\phi|^2)}{2} & j_r &= -\frac{\Pi^*\Phi + \Pi\Phi^*}{2a} = a^2j^r \\
S^r_r &= \rho + U(|\phi|^2) & S^\theta_\theta &= \frac{|\Pi|^2 - |\Phi|^2}{2a^2} - \frac{U(|\phi|^2)}{2} \\
S &= \frac{3|\Pi|^2 - |\Phi|^2}{2a^2} - \frac{3}{2}U(|\phi|^2)
\end{aligned}$$

EOM for complex scalar field with self-interaction potential $U(|\phi|^2)$
Hamiltonian Constraint:

$$R + 4K^r_rK^\theta_\theta + 2{K^\theta_\theta}^2 = 8\pi\rho$$

$$\frac{-2}{arb}\left\{\left[\frac{(rb)'}{a}\right]' + \frac{1}{rb}\left[\left(\frac{rb}{a}(rb)'\right)' - a\right]\right\} + 4K^r_rK^\theta_\theta + 2K^\theta_\theta = 8\pi\left[\frac{|\Phi|^2 + |\Pi|^2}{a^2} + U(|\phi|^2)\right]$$

Momentum constraint:

$$K_\theta^{\theta\prime} + \frac{(rb)'}{rb}\left(K_\theta^\theta - K_r^r\right) = 2\pi\frac{\Pi^*\Phi + \Pi\Phi^*}{a} \tag{34}$$

Evolution equations for the metric (γ_{ij}) functions

$$\dot{a} = -\alpha a K_r^r + (a\beta)' \tag{35}$$

$$\dot{b} = -\alpha b K_\theta^\theta + \frac{\beta}{r}(rb)' \tag{36}$$

Evolution equations for the extrinsic curvature K_j^i

$$\dot{K}_r^r = \beta K_r^{r\prime} - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left\{\frac{-2}{arb}\left[\frac{(rb)'}{a}\right]' + KK_r^r - 4\pi\left[\frac{2|\Phi|^2}{a^2} + U(|\phi|^2)\right]\right\}$$

$$\dot{K}_\theta^\theta = \beta K_\theta^{\theta\prime} + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left[\frac{\alpha rb}{a}(rb)'\right]' + \alpha K K_\theta^\theta - 4\pi\alpha U(|\phi|^2)$$

Evolution equations for the scalar field and auxiliary fields

$$\dot{\phi} = \frac{\alpha}{a}\Pi + \beta\Phi \tag{37}$$

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi + \beta\Phi\right)' \tag{38}$$

$$\dot{\Pi} = \frac{1}{(rb)^2}\left[(rb)^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right]' + 2\left[\alpha K_\theta^\theta - \beta\frac{(rb)'}{rb}\right]\Pi - \alpha a\frac{dU(|\phi|^2)}{d|\phi|^2}\phi \tag{39}$$

Initial value problem in spherical symmetry polar-areal coordinates
Our ansatz is

$$\phi(r,t) = \phi_0(r)e^{-i\omega t}, \tag{40}$$

that is, we have a time dependent field producing a static metric Static spacetime: $\beta = 0$, $\dot{a} = 0 = \dot{b}$ then, from equations 35 and 36 we get that

$$K_r^r = 0 \quad \text{and} \quad K_\theta^\theta = 0 \tag{41}$$

From the ansatz in equation 40, it follows that

$$\dot{\phi} = -i\omega\phi_0(r)e^{i\omega t}\phi' = \phi_0' e^{-i\omega t} \tag{42}$$

from which we find that

$$\Pi = \frac{-i\omega a}{\alpha}\phi_0 e^{-i\omega t}\Phi = \phi_0' e^{-i\omega t} \tag{43}$$

With the ansatz and static spacetime, the momentum constraint is satisfied for all time ie $\Pi^*\Phi + \Phi^*\Pi = 0$

$$K_\theta^{\theta\prime} + \frac{(rb)'}{rb}(K_\theta^\theta - K_r^r) = \frac{2\pi}{a}(\Pi^*\Phi + \Pi\Phi^*) \tag{44}$$

The two parenthesis evaluate to zero, yielding

$$K_\theta^{\theta\prime} = 0 \Longrightarrow \quad K_\theta^\theta = \text{constant} = 0 \tag{45}$$

Polar-areal:
We have the slicing condition $K = K_r^r$, the spatial coordinate condition $b = 1$, the boson star ansatz, timelike Killing vectors ($\beta = 0$) and a static spacetime ($\dot{a} = \dot{b} = 0$). Then the metric becomes

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 \tag{46}$$

From the slicing condition we see that

$$K_r^r + 2K_\theta^\theta = K_r^r \implies K_\theta^\theta \tag{47}$$

so, again, we have that

$$K_r^r = 0 \quad \text{and} \quad K_\theta^\theta = 0 \tag{48}$$

to enforce polar slicings all the time, we, furthermore, require that $K_\theta^\theta = 0$
Hamiltonian Constraint:

$$\begin{aligned}
&\frac{-2}{arb}\left\{\left[\frac{(rb)'}{a}\right]' + \frac{1}{rb}\left[\left(\frac{(rb)'}{a}(rb)'\right)' - a\right]\right\} - 4K_r^r K_\theta^\theta + 2{K_\theta^\theta}^2 = 8\pi\left[\frac{|\Phi|^2 + |\Pi|^2}{a^2} + U(|\phi|^2)\right] \\
&\implies \frac{-2}{ar}\left\{\left(\frac{1}{a}\right)' + \frac{1}{r}\left[\left(\frac{r}{a}\right)' - a\right]\right\} = 8\pi\left[\frac{\Phi\Phi^* + \Pi\Pi^*}{a^2} + U(|\phi|^2)\right] \\
&\implies \frac{-2}{ar}\left\{-\frac{a'}{a^2} + \frac{1}{r}\left[\frac{1}{a} - \frac{a'r}{a^2} - a\right]\right\} = 8\pi\left[\frac{{\phi_0'}^2 + \frac{a^2\omega^2}{\alpha^2}\phi_0^2}{a^2} + U(|\phi|^2)\right] \\
&\implies \frac{-1}{ar}\left\{-a' + \frac{1}{r}\left[a - a'r - a^3\right]\right\} = 8\pi\left[\Phi_0^2 + \frac{a^2\omega^2}{\alpha^2}\phi_0^2 + a^2U(|\phi|^2)\right] \\
&\implies -2a' + \frac{a}{r}\left[1 - a^2\right] = 4\pi ar\left[\Phi_0^2 + \frac{a^2\omega^2}{\alpha^2}\phi_0^2 + a^2U(|\phi|^2)\right] \\
&\implies a' = \frac{1}{2}\left\{\frac{a}{r}\left[1 - a^2\right] + 4\pi ar\left[\Phi_0^2 + \phi_0^2 a^2\left(\frac{\omega^2}{\alpha^2} + \frac{U(|\phi|^2)}{\phi_0^2}\right)\right]\right\} \\
&\implies a' = \frac{1}{2}\left\{\frac{a}{r}\left[1 - a^2\right] + 4\pi ar\left[\Phi_0^2 + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + U(|\phi|^2)a^2\right]\right\}
\end{aligned} \tag{49}$$

An example for a potential would be what is interpreted as the mass term $U(|\phi|^2) = m^2|\phi|^2 = m^2\phi_0^2$.

Solution of K^θ_θ (Applying slicing condition, BS ansatz and static spacetime) in polar coordinates.

$$
\begin{aligned}
K^\theta_\theta &= \beta K^{\theta\prime}_\theta + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left[\frac{\alpha rb}{a}(rb)'\right]' + \alpha K K^\theta_\theta - 4\pi\alpha U(|\phi|^2) \\
&\implies 0 = \frac{\alpha}{r^2} - \frac{1}{ar^2}\left[\frac{\alpha r}{a}\right]' - 4\pi\alpha U(|\phi|^2) \\
&\implies 0 = \frac{\alpha}{r^2} - \frac{1}{ar^2}\left[\left(\frac{\alpha'}{a} - \frac{\alpha a'}{a^2}\right)' r + \frac{\alpha}{a}\right] - 4\pi\alpha U(|\phi|^2) \\
&\implies 0 = a\alpha - \left\{\left[\frac{\alpha' r}{a} - \frac{r\alpha}{2a^2}\left(\frac{a}{r}(1-a^2) + 4\pi ar\left(a^2U(\phi_0^2) + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right)\right)\right] + \frac{\alpha}{a}\right\} - 4\pi\alpha r^2 U(|\phi|^2) \\
&\implies 0 = a\alpha - \left\{\frac{\alpha' r}{a} - \frac{\alpha}{2a}(1-a^2) - \frac{2\pi r^2}{a}\alpha\left(a^2U(\phi_0^2) + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right)\frac{\alpha}{a}\right\} - \frac{4\pi\alpha r^2}{a}U(|\phi|^2) \\
&\implies 0 = a\alpha - \frac{\alpha' r}{a} + \frac{\alpha}{2a} - \frac{\alpha a}{2} + 2\pi r^2\alpha a U(\phi_0^2) + \frac{2\pi r^2\alpha}{a}\left(\frac{\omega^2}{\alpha^2}\Phi_0^2 a^2\right) - \frac{\alpha}{a} - 4\pi\alpha a r^2 U(|\phi|^2) \\
&\implies 0 = \frac{a\alpha}{2} - \frac{\alpha' r}{a} - \frac{\alpha}{2a} + 2\pi r^2\alpha a U(\phi_0^2) + \frac{2\pi r^2\alpha}{a}\left(\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right) \\
&\implies \alpha' = \frac{\alpha}{2r}(a^2-1) + 2\pi r\alpha\left(-a^2U(\phi_0^2) + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right)
\end{aligned}
$$

(50)

$$
\implies \alpha' = \frac{\alpha}{2}\left\{\frac{1}{r}(a^2-1) + 4\pi r\left(\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 - a^2U(\phi_0^2) + \Phi_0^2\right)\right\}
$$

(51)
$$\phi' = \Phi$$

From K-G-si equations:

(52)
$$\nabla^\mu\nabla_\nu\phi = \frac{dU(|\phi|^2)}{d|\phi|^2}\phi$$

From equations 37, 38 and 39 we see that

$$
\begin{aligned}
\dot\phi &= \frac{\alpha}{a}\Pi + \beta\Phi \implies -i\omega\phi_0 e^{-i\omega t} = -\frac{\alpha}{a}i\omega\frac{a}{\alpha}\phi_0 e^{-i\omega t} \\
\dot\Phi &= \left(\frac{\alpha}{a}\Pi\right)' \implies -\phi_0' i\omega e^{-i\omega t} = \left(-\frac{\alpha}{a}\frac{i\omega a}{\alpha}\phi_0 e^{-i\omega t}\right) \\
\dot\Pi &= \frac{1}{(rb)^2}\left[(rb)^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right]' + 2\left[\alpha K^\theta_\theta - \frac{\beta(rb)'}{rb}\right]\Pi - \alpha a\frac{dU(|\phi_0|^2)}{d|\phi_0|^2}\phi
\end{aligned}
$$

From 43

(53)
$$\dot\Pi = -\frac{\omega^2 a}{\alpha}\phi_0 e^{-i\omega t}$$

which gives us

$$\begin{aligned}
\frac{\omega^2 a}{\alpha}\phi_0 e^{-i\omega t} &= \frac{1}{r^2}\left[\frac{\alpha r^2}{a}\Phi\right]' - \alpha a \frac{dU(\phi_0^2)}{d\phi_0}\phi_0 e^{-i\omega t} \\
\implies \frac{\omega^2 a}{\alpha}\phi_0 &= \frac{1}{r^2}\left[\frac{\alpha r^2}{a}\Phi_0\right]' - \alpha a \frac{dU(\phi_0^2)}{d\phi_0}\phi_0 \\
\implies \alpha a \frac{dU(\phi_0^2)}{d\phi_0^2}\phi_0 - \frac{\omega^2 a r^2}{\alpha}\phi_0 &= \left(\frac{alpha}{a}\right)' r^2\Phi_0 + \frac{\alpha}{a}2r\Phi_0 + \frac{\alpha r^2}{a}\Phi_0' \\
\implies \frac{\alpha a}{r^2}\frac{dU(\phi_0^2)}{d\phi_0^2}\phi_0 - \frac{\omega^2 a}{\alpha}\phi_0 &= \left[\left(\frac{alpha}{a}\right)' + \frac{2\alpha}{ar}\right]\Phi_0 + \frac{\alpha}{a}\Phi_0'
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\alpha}{a}\right)' + \frac{2}{r} &= \frac{\alpha'}{a} - \frac{\alpha a'}{a^2} + \frac{2\alpha}{ra} \\
&= \frac{\alpha}{2a}\left\{\frac{1}{r}(a^2-1) + 4\pi r\left(\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 - a^2 U(\phi_0^2) + \Phi_0^2\right)\right\} \\
&- \frac{\alpha}{2a}\left\{\frac{1}{r}(a^2-1) + 4\pi r\left(\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + a^2 U(\phi_0^2) + \Phi_0^2\right)\right\} \\
&+ \frac{2\alpha}{ra} \\
&= \frac{\alpha}{a}\frac{a^2-1}{r} - \frac{\alpha}{a}4\pi r a^2 U(\phi_0^2) + \frac{2\alpha}{ar} \\
&= \frac{\alpha(a^2+1)}{ar} - \alpha 4\pi r a U(\phi_0^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{a}\Phi_0' &= \alpha a \frac{dU(\phi_0^2)}{d\phi_0^2}\phi_0 - \frac{\omega^2 a}{\alpha}\phi_0 - \left[\frac{\alpha}{ar}(a^2+1) - \alpha 4\pi r a U(\phi_0^2)\right]\Phi_0 \\
\iff \Phi_0' &= a^2\frac{dU(\phi_0^2)}{d\phi_0^2}\phi_0 - \frac{\omega^2 a^2}{\alpha^2}\phi_0 - \left[1 + a^2 - \alpha 4\pi r^2 a^2 U(\phi_0^2)\right]\frac{\Phi_0}{r} \\
\iff \Phi_0' &= \left(\frac{dU(\phi_0^2)}{d\phi_0^2} + \frac{\omega^2}{\alpha^2}\right)a^2\phi_0 - \left[1 + a^2 - \alpha 4\pi r^2 a^2 U(\phi_0^2)\right]\frac{\Phi_0}{r} \qquad (54)
\end{aligned}$$

To summarize: The set of equatins to be integrated ar then

$$a' = \frac{1}{2}\left\{\frac{a}{r}(1-a^2) + 4\pi a r\left[a^2 U(\phi_0^2) + \frac{\omega^2}{\alpha^2}\phi_0^2 a^2 + \Phi_0^2\right]\right\} \qquad (49)$$

$$\alpha' = \frac{\alpha}{2}\left\{\frac{1}{r}(a^2-1) + 4\pi a r\left[\frac{\omega^2}{\alpha^2}\phi_0^2 a^2 - a^2 U(\phi_0^2) + \Phi_0^2\right]\right\} \qquad (50)$$

$$\phi_0' = \Phi_0 \qquad (51)$$

$$\Phi_0' = \left(\frac{dU(\phi_0^2)}{d\phi_0^2} - \frac{\omega^2}{\alpha^2}\right)a^2\phi_0 - \left(1 + a^2 - 4\pi r^2 a^2 U(\phi_0^2)\right)\frac{\Phi_0}{r} \qquad (54)$$

Regularity and boundary conditions: In order to make equation 49 regular at the origin the limit

$$\lim_{r\to 0}\frac{a}{r}(1-a^2) = \text{const.} \qquad (55)$$

One way to achive this is to set $a(0) = 0$ or $a(0) = \pm 1$. The first regularity contidion should be ruled out since we want all the sperically symmetric metric functions to be positive. We choose to use $a(0) = 1$
A regular function at the origin has the following series expansin:

$$\lim_{r \to 0} a(r) = a_0 + a_2 r^2 + \ldots \implies a(0) = a_0 = 1 \tag{56}$$

Then

$$a(0) = 1 \qquad \Phi_0(0) = \phi_0'(0) \tag{57}$$

and the outer boundary condition for the field becomes

$$\lim_{r \to \infty} \phi_0(r) = 0 \tag{58}$$

For the b.c. for α, we need to evaluate the mass aspect function, which is done in the next section (see equation 61).
Mass aspect function
Polar-areal metric ($b = 1$, $\beta = 0$):

$$ds^2 = -\alpha^2 dT^2 + a^2 dR^2 + R^2 d\Omega^2 \tag{59}$$

When $R \to \infty$ we expect the metric to approach the usual Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \tag{60}$$

We define the mass aspect function such that

$$\left(1 - \frac{2M(T,R)}{R}\right)^{-1} \equiv a^2(T,R) \qquad \implies \qquad M(T,R) = \frac{R}{2}\left(1 - \frac{1}{a^2(T,R)}\right)$$

The outer boundary for $\alpha(T,R)$ can be derived by noticing that:

$$\begin{aligned} \lim_{R \to \infty} \alpha^2(T,R) &\equiv \left(1 - \frac{2M}{R}\right)\frac{1}{a^2} \\ \implies \lim_{R \to \infty} \alpha(T,R) &= \frac{1}{a(T,R)} \end{aligned} \tag{61}$$

As equation 50 is linear in α we can convert the outer boundary condition, 61, for α int an inner one. After choosing $\alpha(0) = 1$ and integrating equations 49, 50, 51 and 54, we have to rescale α as well as ω so that

$$\alpha(R_{\max}) \to c\alpha(R_{\max}) = \frac{1}{a(R_{\max})} \implies c = \frac{1}{\alpha(R_{\max})a(R_{\max})} \tag{62}$$

In the equations 49 to 54 it's clear tat we have to resccale ω since

$$\frac{\omega^2}{\alpha^2} \to \frac{c^2\omega^2}{c^2\alpha^2} = \frac{\omega^2}{\alpha^2} \tag{63}$$

Thus, it won't change the eigenvalue problem!

$$\begin{aligned} \omega &\to c\omega \\ \alpha &\to c\alpha \end{aligned}$$