PHY 387N: Relativity Theory II Spring 1998 Project 3
Due: Thursday, April 9, 12:30pm

Problem 1: *Critical (Black-Hole-Threshold) Behaviour in Yang-Mills Collapse*

1.1) *Preamble:*

In this project you will study the spherically symmetric gravitational collapse of an SU(2) Yang-Mills field as described in Choptuik, Chmaj and Bizon (hereafter CCB), "Critical Behaviour in Gravitational Collapse of a Yang-Mills Field", *Phys. Rev. Lett.* **77**, 424-427 (1996), also available on-line in essentially identical form at `http://xxx.lanl.gov/ps/gr-qc/9603051`. This model exhibits two types of black-hole threshold phenomena, dubbed Type I and Type II behaviour, respectively. In both cases, the threshold of black hole formation is characterized by a special, and essentially unique, *critical solution* of the EYM (Einstein-Yang-Mills) equations, which can be generated dynamically via fine-tuning of initial data. It is important to realize that these critical solutions are *unstable*: for *generic* (i.e. non-tuned) initial data, in the asymptotic (i.e. infinite time) limit, either all of the matter will have escaped to infinity, or some percentage of the matter will have escaped and the rest will have been trapped in a black hole. By fine-tuning the initial conditions we can exploit this natural "competition" to produce critical solutions which, if infinitesimally perturbed, result in either complete dispersal or black hole formation.

The Type I critical solution is *static*, and was first discovered by Bartnik and McKinnon (*PRL*, **61**, 141-144 (1988), who adopted a static *ansatz* for the model and then, solving the resulting set of ODEs via a shooting technique, generated convincing evidence for a countable infinity of regular solutions labeled by an integer, n, which counts the number of zero-crossings of the Yang-Mills potential $W(r,t)$ (see below). Specifically the Type I critical solution you are to study is precisely the $n = 1$ solution found by those authors. If this solution is perturbed in a way which leads to collapse, rather than dispersal, a black hole with *finite* mass forms. Viewing the mass as an order parameter, this is analogous to a first order phase transition in statistical mechanics—hence the nomenclature "Type I".

The Type II critical solution in the model is very much analogous to the original scalar field critical solution (Choptuik, *PRL*, **70**, 9–12 (1993)), in that it is (discretely) self-similar (so rather than having a temporal Killing vector, as in the static Type I case, we have a "scale Killing vector") and consequently is strong-field to arbitrarily small spatio-temporal scales and contains a naked singularity. In this case, black hole formation turns on at infinitesimal mass (analogous to a second order phase transition) and the masses of the black holes which form are well-fit by a scaling law

$$M_{BH} \propto |p - p^\star|^\gamma \,, \tag{1}$$

where p is the tuning parameter, $p^\star$ is the critical (threshold value), and $\gamma \approx 0.20$ is a *universal* (i.e. initial data-independent) scaling exponent. Due to the self-similar nature of the Type II solution, a detailed finite-difference-based study requires adaptive mesh refinement, and it is primarily for this reason that we will focus on the Type I behaviour. However, if you are interested, and if you have time, you are encouraged to use your code to investigate the Type II behaviour in the model to whatever extent is feasible (*optional* Problem 1f).

Finally note that the specific requirements for "completion" of this project are purposefully being left somewhat vague, primarily since telling you *exactly* what to do, and how to do it, is obviously highly artificial in terms of the "real–world" arena of computationally-oriented research. Also, as I have emphasized several times in the course, you want to get your code going and tested as quickly as possible so that you are left with as much time as possible to explore and analyze the phenomenology exhibited by the model.

1.2) *Equations of Motion:*

A full motivation and description of the physics and mathematics underlying this model is (a) beyond the scope of this project, and (b) not necessary in order for you to solve the equations of motion and observe the basic phenomenology. Those interested in the details should consult the CCB paper and references contained therein. In brief, by combining spherical symmetry with a certain additional *ansatz* for the the SU(2) Yang-Mills field (purely magnetic *ansatz*, Abelian gauge) *and adopting polar/areal (PA) coordinates*, we can write down a simple Lagrangian for the model which is very similar in form to that for a single scalar field with a non-trivial self-interaction, coupled to gravity.

Specifically, the space-time metric is

$$ds^2 = -\alpha^2(r,t)\,dt^2 + a^2(r,t)\,dr^2 + r^2 d\Omega^2\,, \tag{2}$$

(PA coordinates), and the matter content is described by a single function, $W \equiv W(r,t)$—which we will call the Yang-Mills potential—with a Lagrangian scalar, L_M:

$$L_M = -\left(\frac{g^{\mu\nu}\,\nabla_\mu W\,\nabla_\nu W}{r^2} + \frac{1}{2}\frac{\left(1-W^2\right)^2}{r^4}\right)\,. \tag{3}$$

It must be stressed that L_M should be viewed as valid *only* within the context of our particular coordinate system—i.e. (3) will generate the correct equations of motion in PA coordinates, but (naive) extensions to the case of other coordinate systems may not yield the appropriate Lagrangian.

Recall that the minimal coupling procedure dictates that the total Lagrangian density, $\mathcal{L}$, is

$$\mathcal{L} = \mathcal{L}_G + \alpha_M \mathcal{L}_M = \sqrt{-g}\,\left(R + \alpha_M L_M\right)\,, \tag{4}$$

where L_M is the matter Lagrangian *scalar* and α_M is some coupling constant with arbitrary magnitude. If we now define the stress tensor, $T_{\mu\nu}$, via

$$T_{\mu\nu} = \frac{\alpha_M}{8\pi}\left(-\frac{\partial L_M}{\partial g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} L_M\right)\,, \tag{5}$$

then extremization of the action with respect to variations of $g^{\mu\nu}$ yields the Einstein fields equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{6}$$

while variation with respect to the matter fields produces the matter equations of motion. Also recall that the definition (5) guarantees, by virtue of the contracted Bianchi identity, that $T_{\mu\nu}$ is conserved

$$\nabla^\mu G_{\mu\nu} = 0 \longrightarrow \nabla^\mu T_{\mu\nu} = 0\,. \tag{7}$$

Problem 1a): Setting the coupling constant

$$\alpha_M = 4\,, \tag{8}$$

derive the following equations of motion for the Yang-Mills potential and the geometric variables:

$$\dot\Phi = \left(\frac{\alpha}{a}\Pi\right)'\,, \tag{9}$$

$$\dot\Pi = \left(\frac{\alpha}{a}\Phi\right)' + \frac{\alpha a}{r^2}\,W\left(1-W^2\right)\,, \tag{10}$$

$$\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}\left(1-W^2\right)^2\right) = 0\,, \tag{11}$$

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}\left(1-W^2\right)^2\right) = 0\,, \tag{12}$$

$$\dot a = \frac{2\alpha}{r}\,\Pi\Phi\,, \tag{13}$$

where

$$
\begin{aligned}
\Phi &\equiv W' , &(14)\\
\Pi &\equiv \frac{a}{\alpha}\dot{W} , &(15)
\end{aligned}
$$

and $W(r,t)$ is to be regarded as a "derived" quantity:

$$W(r,t) = W_0 + \int_0^r \Phi(\tilde{r},t)\, d\tilde{r} . \tag{16}$$

As usual, overdots and primes denote partial differential differentiation with respect to time and space, respectively. Note that (11) is the Hamiltonian constraint, (12) is the polar slicing condition, and (13) is the evolution equation for a, wherein the momentum constraint has been used to eliminate the extrinsic curvature component, $K^r{}_r$. In deriving (11-13), you may find it useful to first derive the non-vanishing stress-tensor components $T_{\mu\nu}$ from (5), then compute the 3+1 quantities

$$
\begin{aligned}
\rho &= n^\mu n^\nu T_{\mu\nu} , &(17)\\
j_i &= -n_\mu T^\mu{}_i , &(18)\\
S^i{}_j &= \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj} , &(19)
\end{aligned}
$$

and, finally, use the appropriate equations discussed in class for the case of a *general* matter source in PA coordinates.

Note that we have included the evolution equation (13) for a for completeness, and for use in the consistency check described below. It should *not* be used to update a in your dynamical evolutions of the model.

Problem 1b): Perform the following non-trivial consistency check of equations (9–15): Solve the Hamiltonian constraint for a', differentiate the result with respect to time and show that one gets the same result by differentiating the right hand side of (13) with respect to r. In showing this equality of $(\dot{a})'$ and $(\dot{a'})$, you will want to eliminate a', α', $\dot{\Phi}$, $\dot{\Pi}$, $\dot{W}$ and W'—wherever they appear—using the full set of equations (9–15). *Hint:* Although it is certainly possible to do this check by hand, it is clearly a job ideally suited for a symbolic manipulation package such as `Maple`.

1.3) *Vacuum States and Regularity Conditions:*

In contrast to the case of a massless scalar field, where $\phi(r,t)$ and $\phi(r,t)+k$, for arbitrary constant k, are identical solutions physically, and where *any* constant solution

$$\phi(r,t) = \text{constant} , \tag{20}$$

is a quiescent, or vacuum, solution, the Yang-Mills matter field has precisely two (discrete) vacuum states:

$$W(r,t) = \pm 1 . \tag{21}$$

This fact is at least partly responsible for much of the interesting phenomenology in the model, including the existence of the Bartnik-McKinnon static solutions. During an evolution, we demand that W remain in specific vacuum states both at $r=0$ and at $r\to\infty$. Without loss of generality, we can set (see (16))

$$W(0,t) \equiv W_0 = +1 , \tag{22}$$

but then we can have either

$$\lim_{r\to\infty} W(r,t) = +1 , \tag{23}$$

or

$$\lim_{r\to\infty} W(r,t) = -1 . \tag{24}$$

It can be shown that regularity of the Yang-Mills field at the origin requires

$$\lim_{r\to 0} W(r,t) = W_0 + r^2 W_2(t) + O(r^4) . \tag{25}$$

(note that W_0 is a *constant*, not a function of t), while the regularity/local-flatness conditions on the geometric variables at $r = 0$ are the usual ones:

$$\lim_{r\to 0} a(r,t) = a_0(t) + r^2 a_2(t) + O(r^4) = 1 + r^2 a_2(t) + O(r^4)\,, \tag{26}$$

$$\lim_{r\to 0} \alpha(r,t) = \alpha_0(t) + r^2 \alpha_2(t) + O(r^4)\,, \tag{27}$$

or equivalently

$$a(0,t) = 1\,, \tag{28}$$

$$a'(0,t) = 0\,, \tag{29}$$

$$\alpha'(0,t) = 0\,. \tag{30}$$

1.4) *Boundary and Initial Conditions*

Examination of the structure of equations (9–13) shows that we can set up initial data for an evolution by specifying (a) $W(r,0)$ (subject to the conditions that $W(0,0) = 1$, $W(\infty,0) = \pm 1$)—from which we can immediately compute $\Phi(r,0)$—and (b) $\Pi(r,0)$.

Problem 1c): Derive expressions for $\Phi(r,0)$ and $\Pi(r,0)$ in terms of

$$W_0(r) \equiv W(r,0)\,, \tag{31}$$

and derivatives of W_0, so that, as much as possible, the Yang-Mills field is initially in-going only. Also derive approximate outgoing radiation boundary conditions for $\Phi(r,t)$ and $\Pi(r,t)$. When solving the slicing constraint, use a boundary condition based on the requirement:

$$\lim_{r\to\infty} \alpha(r,t) = \frac{1}{a(r,t)}\,, \tag{32}$$

which, as discussed in class, follows from identification of (2) with the usual (static) Schwarzschild line element (Birkhoff's theorem), and the demand that t measure proper time of coordinate stationary observers as $r \to \infty$.

1.5) *Solution of the Equations of Motion*

Use RNPL to generate an $O(h^2)$ finite-difference code called `eym` to solve (9-12,16) with boundary and initial conditions as described above. Your code should employ the following techniques and features:

1. Crank-Nicholson differencing—with implicit Kreiss-Oliger-style dissipation—of equations (9) and (10) (see supplementary notes for descriptions of these techniques and the online project resources for an example).
2. Approximate outgoing radiation conditions.
3. Hand-coded solvers for the Hamiltonian constraint (11) and the slicing condition (12), incorporated into RNPL using the `UPDATE` statement (see my RNPL solution of the EMKG model, available online, for an example—I encourage you to write your own solvers "from scratch", but you can also borrow code "wholesale" from my example if you so desire).
4. Bullet proofing of the Hamiltonian constraint solver, so that if the solver fails, the program exits gracefully with an appropriate error message.
5. Computation of the "black hole" function, $Z(r,t)$

   $$Z(r,t) \equiv \frac{2m(r,t)}{r}\,, \tag{33}$$

   where $m(r,t)$ is the usual PR mass-aspect function:

   $$m = \frac{1}{2} r \left(1 - a^{-2}\right)\,. \tag{34}$$

6. Monitoring of the instantaneous maximum of $Z(r,t)$ with a graceful program exit when a user-specified threshold value (presumably indicating imminent black hole formation) is exceeded during an evolution—you will need to determine an appropriate value for the threshold empirically.

7. Flat-spacetime option: Use an RNPL parameter `flatspace`, which, if set to a non-zero value, results in

$$a(r,t) = \alpha(r,t) = 1\,, \tag{35}$$

so that the propagation of the YM potential in flat-spacetime can also be studied with your program.

You may find it more convenient to incorporate your computation of m, Z etc. into your constraint-solving code, rather than having RNPL generate the code for you. As usual, check your code carefully for convergence and for physically reasonable behaviour—you can be sure that I will do the same!

1.6) *Critical Behaviour*

Consider the following initial data

$$W_0(r; r_0, \delta) = \frac{1 + \left(r_0^2 - r^2\right)/\delta^2}{\left(\left(1 + \left(r_0^2 - r^2\right)/\delta^2\right)^2 + 4r^2\right)^{1/2}}\,, \tag{36}$$

where r_0 and δ are adjustable parameters. Provided that $0 << r_0 <<$ `rmax`, where `rmax` is the radius at the outer edge of the computational domain, this data describes a "kink", centred at $r = r_0$, with a "kink-width" controlled by δ. As δ decreases, the kink steepens, and the configuration becomes more strongly self-gravitating. Note that $\Pi(r, 0)$ is to be specified (as you computed above), so that the kink is, as much as possible, initially in-going.

Problem 1d): Use your code with the above initial data to study the Type I critical behaviour in the model. In your write-up, be sure to include a description of your basic methodology, (e.g. how you found the critical solution) along with qualitative and, where possible, quantitative discussions of your findings. Make some `.mpeg` or `.gif` movies showing key features and post them on your web page.

Note: If you have time, you may want to write an LSODA-based program to compute the $n = 1$ Bartnik-McKinnon solution using a shooting technique. However, I will supply the profile in machine-readable form (i.e. in an `.sdf` file). This will allow you to directly verify that you are, indeed, generating the $n = 1$ bartnikon as the critical solution. See the online resources for details.

Problem 1e): Generate the Type I critical solution again, this time using a family of initial data of your *own* design. (No trivial modifications of (36)!).

Problem 1f): *OPTIONAL!:* Study the Type II critical behaviour in the model.

---

2) *SUPPLEMENTARY NOTES*

2.1) *Crank-Nicholson Differencing*

Consider a general first-order-in-time PDE

$$\dot{u}(x,t) = L[u(x,t)]\,, \tag{37}$$

where L is some purely spatial differential operator (such as $\partial_x, \partial_{xxx}$, etc.). Assuming, as usual, a uniform discretization of space and time—$(x,t) \to (x_j, t^n) = (x_0 + j\triangle x, t^0 + n\triangle t) = (x_0 + jh, t^0 + n\lambda h)$, where λ is held fixed as $h \to 0$—we define the *Crank-Nicholson* scheme for (37) as follows:

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\triangle t} = \frac{1}{2}\left(\hat{L}\left[\hat{u}^{n+1}\right]_j + \hat{L}\left[\hat{u}^n\right]_j\right)\,. \tag{38}$$

Further assuming that $\hat{L}$ is an $O(h^2)$ approximation of L, it is easy to see (by considering Taylor-series expansions about $(t^{n+\frac{1}{2}}, x_j)$) that this scheme has $O(h^2)$ truncation error. Note that the term "Crank-Nicholson" (hereafter CN), named for researchers who originally wrote down such a scheme for the diffusion equation, refers (only) to

- The centering of the discrete time derivative at $t = t^{n+\frac{1}{2}}$ using the standard $O(h^2)$ approximation of a first derivative.
- The centering of the discrete spatial differential operators at $t = t^{n+\frac{1}{2}}$ using an averaging procedure.

In particular, note that the nomenclature CN (at least as I'm defining it) implies *nothing* about the specific form of $\hat{L}$, except that we will assume $\hat{L} = L + O(h^2)$.

For example, consider the ordinary wave equation ($' \equiv \partial_x$)

$$\ddot{\phi} = \phi''\,, \tag{39}$$

written in first-order form ($\Phi \equiv \phi'$, $\Pi \equiv \dot{\phi}$):

$$\dot{\Phi} = \Pi'\,, \tag{40}$$

$$\dot{\Pi} = \Phi'\,. \tag{41}$$

Using the usual $O(h^2)$ approximation for ∂_x, a CN form of (40-41) is

$$\frac{\Phi_j^{n+1} - \Phi_j^n}{\triangle t} = \frac{1}{2}\left(\frac{\Pi_{j+1}^{n+1} - \Pi_{j-1}^{n+1}}{2\triangle x} + \frac{\Pi_{j+1}^{n} - \Pi_{j-1}^{n}}{2\triangle x}\right)\,, \tag{42}$$

$$\frac{\Pi_j^{n+1} - \Pi_j^n}{\triangle t} = \frac{1}{2}\left(\frac{\Phi_{j+1}^{n+1} - \Phi_{j-1}^{n+1}}{2\triangle x} + \frac{\Phi_{j+1}^{n} - \Phi_{j-1}^{n}}{2\triangle x}\right)\,. \tag{43}$$

Note that, like all CN methods, this scheme is *implicit*: i.e. we can *not* solve (42)-(43) explicitly for the advanced values Φ_j^{n+1} and Π_j^{n+1}. Although there are more efficient ways of solving such systems, for hyperbolic systems `RNPL`'s built-in iterative update procedure will usually solve such equations quite satisfactorily.

Defining difference operators, Δ_+^t, μ_+^t and Δ_0^x by

$$\Delta_+^t u_j^n \equiv \frac{u_j^{n+1} - u_j^n}{\triangle t}\,, \tag{44}$$

$$\mu_+^t u_j^n \equiv \frac{1}{2}\left(u_j^{n+1} + u_j^n\right)\,, \tag{45}$$

$$\Delta_0^x u_j^n \equiv \frac{u_{j+1}^n - u_{j-1}^n}{2\triangle x}\,, \tag{46}$$

we can rewrite (42)-(43) in a form which is readily translated into RNPL:

$$\Delta_+^t \Phi = \mu_+^t \left(\Delta_0^x \Pi\right) , \quad (47)$$

$$\Delta_+^t \Pi = \mu_+^t \left(\Delta_0^x \Phi\right) . \quad (48)$$

At least for simple hyperbolic systems such as (39), one also finds that outgoing radiation boundary conditions can be incorporated into a CN scheme in the obvious way. Thus, for example, at $x = x_{\max}$, a continuum outgoing condition on Φ is given by:

$$\dot{\Phi} + \Phi' = 0 . \quad (49)$$

Introducing the $O(h^2)$ *backwards* difference approximation of ∂_x, denoted D_-^x:

$$D_-^x = u_j^n = \frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\triangle x} , \quad (50)$$

a suitable approximation of (49) is

$$\Delta_+^t \Phi + \mu_+^t \left(D_-^x \Phi\right) = 0 . \quad (51)$$

2.2) *Implicit "Kreiss-Oliger" Dissipation*

We begin with a general discussion concerning the motivation for the addition of "Kreiss-Oliger" (KO) dissipation terms to finite-difference approximations of hyperbolic systems. In brief, KO dissipation is used to control the *stability* of such difference schemes. Now, the notion of stability is crucially important in FD (finite-difference) solutions of time-dependent problems, and is a subject which we will not be able to discuss in class in any detail. However, the basic ideas are quite easy to understand. (I should also point out at this juncture that is purely from stability considerations that I am suggesting that you use CN differencing rather than leap-frog to solve the EYM equations—you might want to see what happens/goes wrong with leap-frog). We begin by writing our difference scheme in the rather abstract form:

$$\hat{u}^{n+1} = G\left[\hat{u}^n\right] , \quad (52)$$

where we will call G the *update operator*. Note that the discrete approximation, $\hat{u}$, will, in general, be a *vector* of unknowns at each mesh point. Also note that there is no loss in generality in stipulating that our scheme is 2-level (i.e. couples only t^n and t^{n+1} unknowns) since multi-level schemes can be recast in 2-level form via the introduction of auxiliary variables in a fashion completely analogous to the conversion of high-order *differential* equations to first order form. Finally note that (52) encompasses *implicit* schemes as well as implicit ones—a typical implicit scheme can be written as

$$G_1\left[\hat{u}^{n+1}\right] = G_0\left[\hat{u}^n\right] , \quad (53)$$

where G_0 and G_1 are some operators; then assuming G_1 is invertible, we have

$$\hat{u}^{n+1} = (G_1)^{-1} G_0\left[\hat{u}^n\right] \equiv G\left[\hat{u}^n\right] . \quad (54)$$

If we know (as is the case for many hyperbolic systems) that the norm (size) of the continuum solution $u(x,t)$ at any time t is $O(1)$ times the norm of the initial data $u(x,0)$, then in order for this feature to be reflected in the difference solution we must have, roughly speaking, that spectrum of G lie on or within the unit circle in the complex plane. Indeed, by iterating (52), we have

$$\hat{u}^n = G^n\left[\hat{u}^0\right] , \quad (55)$$

where G^n is the n-fold application of the operator G. Clearly then, if the largest eigenvalue ($\sim$ spectral radius) of G has modulus larger than unity, our difference scheme will "blow up" (i.e. be unstable) as we iterate. Unfortunately, this situation occurs all too frequently in practice!

A basic rule of thumb regarding the stability of FD approximations to hyperbolic systems is that is usually the *high-frequency*, or *short-wavelength* components which govern the overall stability of the scheme—i.e.

stability is largely a high-frequency phenomenon. In addition, if one looks at the *dispersion* relation for standard $O(h^2)$ approximations of such systems (such as (42)-(43)), one finds that the phase velocities of high frequency components are usually grossly inaccurate. Thus, Kreiss and Oliger conclude, since high frequency components are potentially troublesome for stability, and tend to be poorly treated by standard diference schemes, it is not at all unreasonable to "go in by hand" and attempt to damp them. This is precisely what KO-style dissipation does.

Consider, for example, the model equation

$$\dot{u} = u' \,, \tag{56}$$

discretized using the $O(h^2)$ leap-frog scheme:

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^{n-1}}{2\triangle t} = \frac{\hat{u}_{j+1}^{n} - \hat{u}_{j-1}^{n}}{2\triangle x} \,. \tag{57}$$

In this case KO dissipation is added by modifying (57) as follows

$$\hat{u}_j^{n-1} \rightarrow \left(1 - \frac{\epsilon}{16}\triangle x^4 D^4\right) \hat{u}_j^{n-1} \,. \tag{58}$$

Here, ϵ is an adjustable *positive* parameter which (somewhat ironically), must satisfy

$$\epsilon < 1 \,, \tag{59}$$

for the modified scheme to be stable. The difference operator D^4 in (58) is defined by

$$D^4 \equiv \left(\Delta_x^+\right)^2 \left(\Delta_-^x\right)^2 \,, \tag{60}$$

$$\Delta_+^x u_j = \frac{u_{j+1} - u_j}{\Delta x} \,, \tag{61}$$

$$\Delta_-^x u_j = \frac{u_j - u_{j-1}}{\Delta x} \,, \tag{62}$$

and is an $O(h^2)$ approximation to ∂_{xxxx}:

$$D^4 u(x,t) = \partial_{xxxx} u(x,t) + O(h^2) \,. \tag{63}$$

Explicitly, we have

$$D^4 u_j^n = \frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{\triangle x^4} \,. \tag{64}$$

The modification (58) is motivated by the following facts:

1. It damps high frequency components effectively—in effect acting as a low-pass filter with quite a sharp cut-off.
2. The difference scheme is stable provided $\lambda \equiv \triangle t/dx < 1$ and $\epsilon < 1$.
3. The difference scheme remains $O(h^2)$ accurate.

Facts (1) and (2) can be established by performing a *Von Neumann stability analysis* of the difference scheme. You can find details concerning this technique in any good textbook on the numerical solution of time-dependent PDEs, but basically, one adopts the "normal mode" *ansatz*:

$$\hat{u}_j^n = \mu^n e^{i\omega j \triangle x} \,, \tag{65}$$

(μ is complex in general), plugs it into the difference scheme, derives a *characteristic equation* which is polynomial in μ

$$P(\mu) = 0 \,, \tag{66}$$

then investigates under what conditions the roots, $\mu_\star$, of (66) lie on or within the unit circle for all values of ω which can be represented on the mesh. In particular, one finds that for high-frequency components (large ω), the corresponding $\mu_\star$'s are well-within the unit circle, and hence will suffer significant damping.

Fact (3) can be seen as follows: from (57) and (58) we have

$$\Delta_0^t \hat{u}_j^n = \Delta_0^x - \frac{\epsilon}{32\triangle t}\triangle x^4\, D^4 \hat{u}_j^n = \Delta_0^x - \frac{\epsilon}{32\lambda}\triangle x^3\, D^4 \hat{u}_j^n\,, \tag{67}$$

so the truncation error of the modified difference scheme is

$$\left(\Delta_0^t - \Delta_0^x + \frac{\epsilon}{32\lambda}\triangle x^3\, D^4\right) u = \left(\Delta_0^t - \Delta_0^x + \frac{\epsilon}{32\lambda}h^3\, D^4\right) u = O(h^2)\,, \tag{68}$$

since from (63)

$$D^4 u = \partial_{xxxx} u + O(h^2) \tag{69}$$

i.e. to leading order, $D^4 u$ is $O(1)$. Thus, KO dissipation modifies the truncation error at the $O(h^3)$ level, so the overall order of the scheme remains unchanged at $O(h^2)$, as claimed.

Finally, the same type of operator can be used to add dissipation to CN (and other implicit schemes). However, we now modify both $\hat{u}_j^{n+1}$ and $\hat{u}_j^n$. Thus, if our CN diff of (56) is

$$\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\triangle t} = \mu_+^t\left(\Delta_o^x \hat{u}_j^n\right) \tag{70}$$

we modify the scheme via (be careful with the signs!)

$$\hat{u}_j^{n+1} \quad\to\quad \left(1 + \frac{\epsilon}{32}\triangle x^4 D^4\right)\hat{u}_j^{n+1}\,, \tag{71}$$

$$\hat{u}_j^{n} \quad\to\quad \left(1 - \frac{\epsilon}{32}\triangle x^4 D^4\right)\hat{u}_j^{n}\,. \tag{72}$$

Explicitly we have

$$\frac{\left(1 + \epsilon\triangle x^4 D^4/32\right)\hat{u}_j^{n+1} - \left(1 - \epsilon\triangle x^4 D^4/32\right)\hat{u}_j^{n}}{\triangle t} = \mu_+^t\left(\Delta_o^x \hat{u}_j^n\right)\,, \tag{73}$$

which can be re-written as

$$\Delta_+^t \hat{u}_j^n = \mu_+^t\left(-\frac{\epsilon}{16\triangle t}\delta^4 \hat{u}_j^n\right) + \mu_+^t\left(\Delta_o^x \hat{u}_j^n\right)\,, \tag{74}$$

where I have defined the fourth-undivided-difference operator, δ^4:

$$\delta^4 u_j^n = \left(\triangle x^4 D^4\right) u_j^n = u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\,. \tag{75}$$

Again, we generally need $\epsilon < 1$ for stability—I typically use $\epsilon = 0.5$, but this is something you'll generally want to experiment with. See the online resources for this project for an example of this technique applied to CN differencing of the wave equation (39).