Lecture Notes on General Relativity

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Abstract

These notes represent approximately one semester's worth of lectures on introductory general relativity for beginning graduate students in physics. Topics include manifolds, Riemannian geometry, Einstein's equations, and three applications: gravitational radiation, black holes, and cosmology. Individual chapters, and potentially updated versions, can be found at http://itp.ucsb.edu/~carroll/notes/.

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Preface

These lectures represent an introductory graduate course in general relativity, both its foundations and applications. They are a lightly edited version of notes I handed out while teaching Physics 8.962, the graduate course in GR at MIT, during the Spring of 1996. Although they are appropriately called "lecture notes", the level of detail is fairly high, either including all necessary steps or leaving gaps that can readily be filled in by the reader. Nevertheless, there are various ways in which these notes differ from a textbook; most importantly, they are not organized into short sections that can be approached in various orders, but are meant to be gone through from start to finish. A special effort has been made to maintain a conversational tone, in an attempt to go slightly beyond the bare results themselves and into the context in which they belong.

The primary question facing any introductory treatment of general relativity is the level of mathematical rigor at which to operate. There is no uniquely proper solution, as different students will respond with different levels of understanding and enthusiasm to different approaches. Recognizing this, I have tried to provide something for everyone. The lectures do not shy away from detailed formalism (as for example in the introduction to manifolds), but also attempt to include concrete examples and informal discussion of the concepts under consideration.

As these are advertised as lecture notes rather than an original text, at times I have shamelessly stolen from various existing books on the subject (especially those by Schutz, Wald, Weinberg, and Misner, Thorne and Wheeler). My philosophy was never to try to seek originality for its own sake; however, originality sometimes crept in just because I thought I could be more clear than existing treatments. None of the substance of the material in these notes is new; the only reason for reading them is if an individual reader finds the explanations here easier to understand than those elsewhere.

Time constraints during the actual semester prevented me from covering some topics in the depth which they deserved, an obvious example being the treatment of cosmology. If the time and motivation come to pass, I may expand and revise the existing notes; updated versions will be available at `http://itp.ucsb.edu/~carroll/notes/`. Of course I will appreciate having my attention drawn to any typographical or scientific errors, as well as suggestions for improvement of all sorts.

Numerous people have contributed greatly both to my own understanding of general relativity and to these notes in particular — too many to acknowledge with any hope of completeness. Special thanks are due to Ted Pyne, who learned the subject along with me, taught me a great deal, and collaborated on a predecessor to this course which we taught as a seminar in the astronomy department at Harvard. Nick Warner taught the graduate course at MIT which I took before ever teaching it, and his notes were (as comparison will reveal) an important influence on these. George Field offered a great deal of advice and encouragement as I learned the subject and struggled to teach it. Tamás Hauer struggled along with me as the teaching assistant for 8.962, and was an invaluable help. All of the students in 8.962 deserve thanks for tolerating my idiosyncrasies and prodding me to ever higher levels of precision.

During the course of writing these notes I was supported by U.S. Dept. of Energy contract no. DE-AC02-76ER03069 and National Science Foundation grants PHY/92-06867 and PHY/94-07195.

Bibliography

The *typical* level of difficulty (especially mathematical) of the books is indicated by a number of asterisks, one meaning mostly introductory and three being advanced. The asterisks are normalized to these lecture notes, which would be given [**]. The first four books were frequently consulted in the preparation of these notes, the next seven are other relativity texts which I have found to be useful, and the last four are mathematical background references.

- B.F. Schutz, *A First Course in General Relativity* (Cambridge, 1985) [*]. This is a very nice introductory text. Especially useful if, for example, you aren't quite clear on what the energy-momentum tensor really means.

- S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972) [**]. A really good book at what it does, especially strong on astrophysics, cosmology, and experimental tests. However, it takes an unusual non-geometric approach to the material, and doesn't discuss black holes.

- C. Misner, K. Thorne and J. Wheeler, *Gravitation* (Freeman, 1973) [**]. A heavy book, in various senses. Most things you want to know are in here, although you might have to work hard to get to them (perhaps learning something unexpected in the process).

- R. Wald, *General Relativity* (Chicago, 1984) [***]. Thorough discussions of a number of advanced topics, including black holes, global structure, and spinors. The approach is more mathematically demanding than the previous books, and the basics are covered pretty quickly.

- E. Taylor and J. Wheeler, *Spacetime Physics* (Freeman, 1992) [*]. A good introduction to special relativity.

- R. D'Inverno, *Introducing Einstein's Relativity* (Oxford, 1992) [**]. A book I haven't looked at very carefully, but it seems as if all the right topics are covered without noticeable ideological distortion.

- A.P. Lightman, W.H. Press, R.H. Price, and S.A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton, 1975) [**]. A sizeable collection of problems in all areas of GR, with fully worked solutions, making it all the more difficult for instructors to invent problems the students can't easily find the answers to.

- N. Straumann, *General Relativity and Relativistic Astrophysics* (Springer-Verlag, 1984) [***]. A fairly high-level book, which starts out with a good deal of abstract geometry and goes on to detailed discussions of stellar structure and other astrophysical topics.

- F. de Felice and C. Clarke, *Relativity on Curved Manifolds* (Cambridge, 1990) [***]. A mathematical approach, but with an excellent emphasis on physically measurable quantities.

- S. Hawking and G. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge, 1973) [***]. An advanced book which emphasizes global techniques and singularity theorems.

- R. Sachs and H. Wu, *General Relativity for Mathematicians* (Springer-Verlag, 1977) [***]. Just what the title says, although the typically dry mathematics prose style is here enlivened by frequent opinionated asides about both physics and mathematics (and the state of the world).

- B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge, 1980) [**]. Another good book by Schutz, this one covering some mathematical points that are left out of the GR book (but at a very accessible level). Included are discussions of Lie derivatives, differential forms, and applications to physics other than GR.

- V. Guillemin and A. Pollack, *Differential Topology* (Prentice-Hall, 1974) [**]. An entertaining survey of manifolds, topology, differential forms, and integration theory.

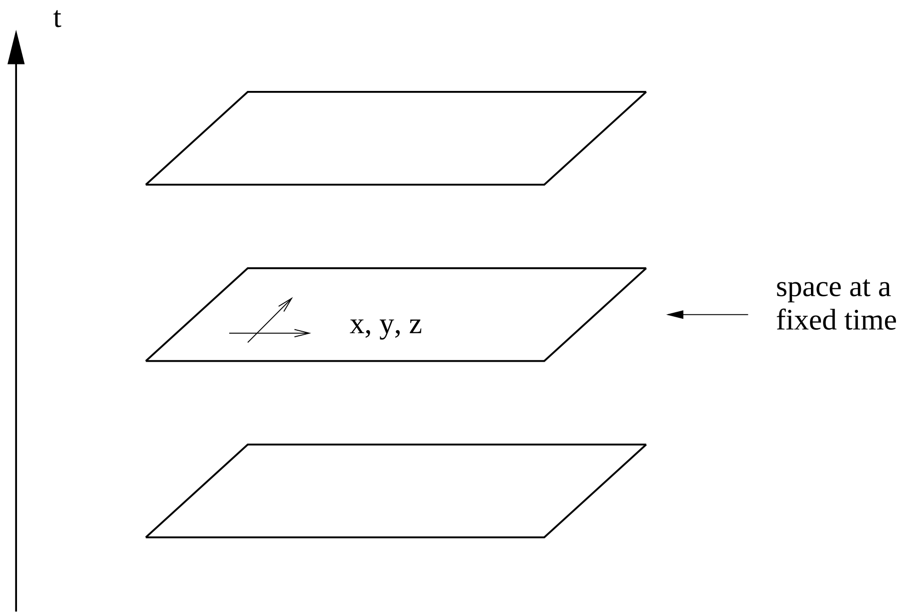
- C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, 1983) [***]. Includes homotopy, homology, fiber bundles and Morse theory, with applications to physics; somewhat concise.

- F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Springer-Verlag, 1983) [***]. The standard text in the field, includes basic topics such as manifolds and tensor fields as well as more advanced subjects.

1 Special Relativity and Flat Spacetime

We will begin with a whirlwind tour of special relativity (SR) and life in flat spacetime. The point will be both to recall what SR is all about, and to introduce tensors and related concepts that will be crucial later on, without the extra complications of curvature on top of everything else. Therefore, for this section we will always be working in flat spacetime, and furthermore we will only use orthonormal (Cartesian-like) coordinates. Needless to say it is possible to do SR in any coordinate system you like, but it turns out that introducing the necessary tools for doing so would take us halfway to curved spaces anyway, so we will put that off for a while.

It is often said that special relativity is a theory of 4-dimensional spacetime: three of space, one of time. But of course, the pre-SR world of Newtonian mechanics featured three spatial dimensions and a time parameter. Nevertheless, there was not much temptation to consider these as different aspects of a single 4-dimensional spacetime. Why not?



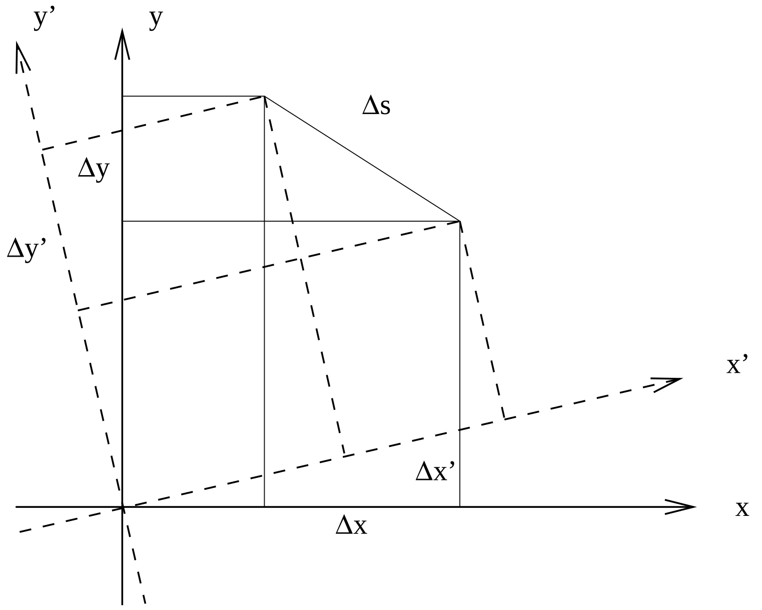
Consider a garden-variety 2-dimensional plane. It is typically convenient to label the points on such a plane by introducing coordinates, for example by defining orthogonal x and y axes and projecting each point onto these axes in the usual way. However, it is clear that most of the interesting geometrical facts about the plane are independent of our choice of coordinates. As a simple example, we can consider the distance between two points, given by

$$s^2 = (\Delta x)^2 + (\Delta y)^2 \ . \tag{1.1}$$

In a different Cartesian coordinate system, defined by x' and y' axes which are rotated with respect to the originals, the formula for the distance is unaltered:

$$s^2 = (\Delta x')^2 + (\Delta y')^2 \ . \tag{1.2}$$

We therefore say that the distance is invariant under such changes of coordinates.



This is why it is useful to think of the plane as 2-dimensional: although we use two distinct numbers to label each point, the numbers are not the essence of the geometry, since we can rotate axes into each other while leaving distances and so forth unchanged. In Newtonian physics this is not the case with space and time; there is no useful notion of rotating space and time into each other. Rather, the notion of "all of space at a single moment in time" has a meaning independent of coordinates.

Such is not the case in SR. Let us consider coordinates (t, x, y, z) on spacetime, set up in the following way. The spatial coordinates (x, y, z) comprise a standard Cartesian system, constructed for example by welding together rigid rods which meet at right angles. The rods must be moving freely, unaccelerated. The time coordinate is defined by a set of clocks which are not moving with respect to the spatial coordinates. (Since this is a thought experiment, we imagine that the rods are infinitely long and there is one clock at every point in space.) The clocks are synchronized in the following sense: if you travel from one point in space to any other in a straight line at constant speed, the time difference between the clocks at the ends of your journey is the same as if you had made the same trip, at the same speed, in the other direction. The coordinate system thus constructed is an **inertial frame**.

An **event** is defined as a single moment in space and time, characterized uniquely by (t, x, y, z). Then, without any motivation for the moment, let us introduce the **spacetime interval** between two events:

$$s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \ . \tag{1.3}$$

(Notice that it can be positive, negative, or zero even for two nonidentical points.) Here, c is some fixed conversion factor between space and time; that is, a fixed velocity. Of course it will turn out to be the speed of light; the important thing, however, is not that photons happen to travel at that speed, but that there exists a c such that the spacetime interval is invariant under changes of coordinates. In other words, if we set up a new inertial frame (t', x', y', z') by repeating our earlier procedure, but allowing for an offset in initial position, angle, and velocity between the new rods and the old, the interval is unchanged:

$$s^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \ . \tag{1.4}$$

This is why it makes sense to think of SR as a theory of 4-dimensional spacetime, known as **Minkowski space**. (This is a special case of a 4-dimensional manifold, which we will deal with in detail later.) As we shall see, the coordinate transformations which we have implicitly defined do, in a sense, rotate space and time into each other. There is no absolute notion of "simultaneous events"; whether two things occur at the same time depends on the coordinates used. Therefore the division of Minkowski space into space and time is a choice we make for our own purposes, not something intrinsic to the situation.

Almost all of the "paradoxes" associated with SR result from a stubborn persistence of the Newtonian notions of a unique time coordinate and the existence of "space at a single moment in time." By thinking in terms of spacetime rather than space and time together, these paradoxes tend to disappear.

Let's introduce some convenient notation. Coordinates on spacetime will be denoted by letters with Greek superscript indices running from 0 to 3, with 0 generally denoting the time coordinate. Thus,

$$x^\mu : \quad \begin{array}{l} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{array} \tag{1.5}$$

(Don't start thinking of the superscripts as exponents.) Furthermore, for the sake of simplicity we will choose units in which

$$c = 1 \ ; \tag{1.6}$$

we will therefore leave out factors of c in all subsequent formulae. Empirically we know that c is the speed of light, 3×10^8 meters per second; thus, we are working in units where 1 second equals 3×10^8 meters. Sometimes it will be useful to refer to the space and time components of x^μ separately, so we will use Latin superscripts to stand for the space components alone:

$$x^i: \quad \begin{array}{l} x^1 = x \\ x^2 = y \\ x^3 = z \end{array} \tag{1.7}$$

It is also convenient to write the spacetime interval in a more compact form. We therefore introduce a 4×4 matrix, the **metric**, which we write using two lower indices:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.8}$$

(Some references, especially field theory books, define the metric with the opposite sign, so be careful.) We then have the nice formula

$$s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu . \tag{1.9}$$

Notice that we use the **summation convention**, in which indices which appear both as superscripts and subscripts are summed over. The content of (1.9) is therefore just the same as (1.3).

Now we can consider coordinate transformations in spacetime at a somewhat more abstract level than before. What kind of transformations leave the interval (1.9) invariant? One simple variety are the translations, which merely shift the coordinates:

$$x^\mu \to x^{\mu'} = x^\mu + a^\mu , \tag{1.10}$$

where a^μ is a set of four fixed numbers. (Notice that we put the prime on the index, not on the x.) Translations leave the differences Δx^μ unchanged, so it is not remarkable that the interval is unchanged. The only other kind of linear transformation is to multiply x^μ by a (spacetime-independent) matrix:

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu , \tag{1.11}$$

or, in more conventional matrix notation,

$$x' = \Lambda x . \tag{1.12}$$

These transformations do not leave the differences Δx^μ unchanged, but multiply them also by the matrix Λ. What kind of matrices will leave the interval invariant? Sticking with the matrix notation, what we would like is

$$\begin{aligned} s^2 = (\Delta x)^{\mathrm{T}}\eta(\Delta x) &= (\Delta x')^{\mathrm{T}}\eta(\Delta x') \\ &= (\Delta x)^{\mathrm{T}}\Lambda^{\mathrm{T}}\eta\Lambda(\Delta x) , \end{aligned} \tag{1.13}$$

and therefore

$$\eta = \Lambda^{\mathrm{T}} \eta \Lambda \ , \tag{1.14}$$

or

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho} \Lambda^{\nu'}{}_{\sigma} \eta_{\mu'\nu'} \ . \tag{1.15}$$

We want to find the matrices $\Lambda^{\mu'}{}_{\nu}$ such that the components of the matrix $\eta_{\mu'\nu'}$ are the same as those of $\eta_{\rho\sigma}$; that is what it means for the interval to be invariant under these transformations.

The matrices which satisfy (1.14) are known as the **Lorentz transformations**; the set of them forms a group under matrix multiplication, known as the **Lorentz group**. There is a close analogy between this group and O(3), the rotation group in three-dimensional space. The rotation group can be thought of as 3×3 matrices R which satisfy

$$\mathbf{1} = R^{\mathrm{T}} \mathbf{1} R \ , \tag{1.16}$$

where $\mathbf{1}$ is the 3×3 identity matrix. The similarity with (1.14) should be clear; the only difference is the minus sign in the first term of the metric η, signifying the timelike direction. The Lorentz group is therefore often referred to as O(3,1). (The 3×3 identity matrix is simply the metric for ordinary flat space. Such a metric, in which all of the eigenvalues are positive, is called **Euclidean**, while those such as (1.8) which feature a single minus sign are called **Lorentzian**.)

Lorentz transformations fall into a number of categories. First there are the conventional **rotations**, such as a rotation in the x-y plane:

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.17}$$

The rotation angle θ is a periodic variable with period 2π. There are also **boosts**, which may be thought of as "rotations between space and time directions." An example is given by

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.18}$$

The boost parameter ϕ, unlike the rotation angle, is defined from $-\infty$ to ∞. There are also discrete transformations which reverse the time direction or one or more of the spatial directions. (When these are excluded we have the proper Lorentz group, SO(3,1).) A general transformation can be obtained by multiplying the individual transformations; the explicit expression for this six-parameter matrix (three boosts, three rotations) is not sufficiently pretty or useful to bother writing down. In general Lorentz transformations will not commute, so the Lorentz group is non-abelian. The set of both translations and Lorentz transformations is a ten-parameter non-abelian group, the **Poincaré group**.

You should not be surprised to learn that the boosts correspond to changing coordinates by moving to a frame which travels at a constant velocity, but let's see it more explicitly. For the transformation given by (1.18), the transformed coordinates t' and x' will be given by

$$\begin{aligned} t' &= t\cosh\phi - x\sinh\phi \\ x' &= -t\sinh\phi + x\cosh\phi \ . \end{aligned} \tag{1.19}$$

From this we see that the point defined by $x' = 0$ is moving; it has a velocity

$$v = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi \ . \tag{1.20}$$

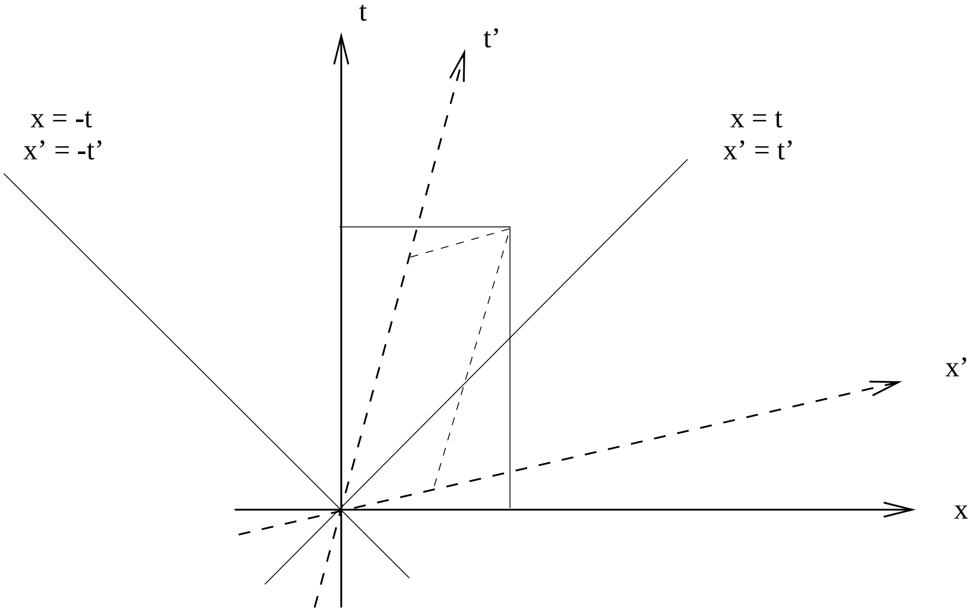
To translate into more pedestrian notation, we can replace $\phi = \tanh^{-1} v$ to obtain

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \end{aligned} \tag{1.21}$$

where $\gamma = 1/\sqrt{1-v^2}$. So indeed, our abstract approach has recovered the conventional expressions for Lorentz transformations. Applying these formulae leads to time dilation, length contraction, and so forth.

An extremely useful tool is the **spacetime diagram**, so let's consider Minkowski space from this point of view. We can begin by portraying the initial t and x axes at (what are conventionally thought of as) right angles, and suppressing the y and z axes. Then according to (1.19), under a boost in the x-t plane the x' axis ($t' = 0$) is given by $t = x\tanh\phi$, while the t' axis ($x' = 0$) is given by $t = x/\tanh\phi$. We therefore see that the space and time axes are rotated into each other, although they scissor together instead of remaining orthogonal in the traditional Euclidean sense. (As we shall see, the axes do in fact remain orthogonal in the Lorentzian sense.) This should come as no surprise, since if spacetime behaved just like a four-dimensional version of space the world would be a very different place.

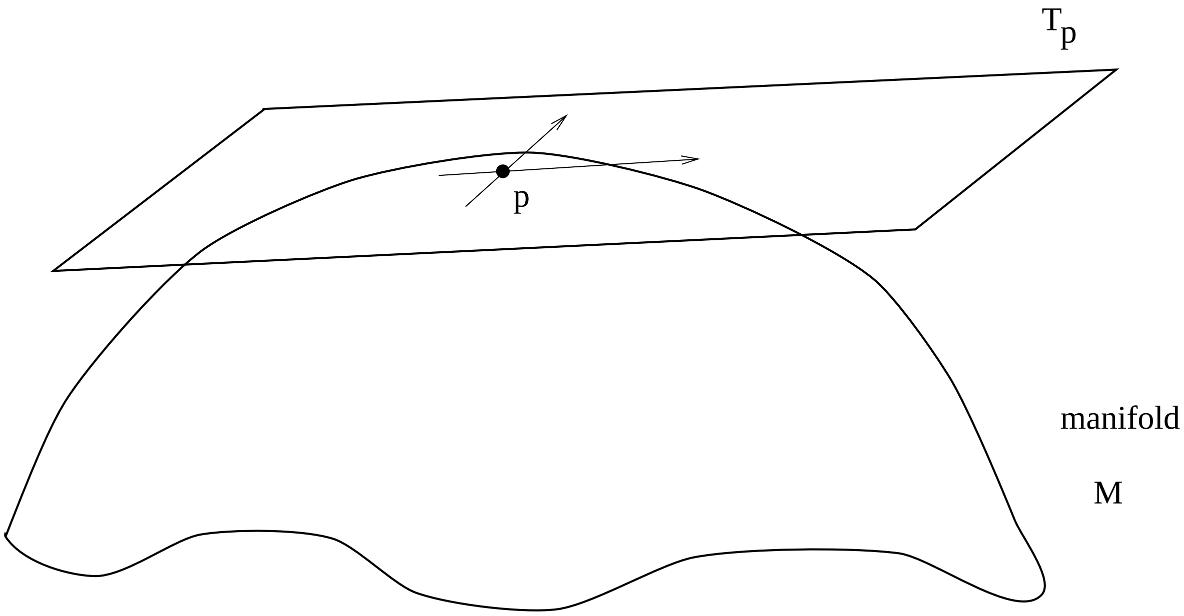
It is also enlightening to consider the paths corresponding to travel at the speed $c = 1$. These are given in the original coordinate system by $x = \pm t$. In the new system, a moment's thought reveals that the paths defined by $x' = \pm t'$ are precisely the same as those defined by $x = \pm t$; these trajectories are left invariant under Lorentz transformations. Of course we know that light travels at this speed; we have therefore found that the speed of light is the same in any inertial frame. A set of points which are all connected to a single event by



straight lines moving at the speed of light is called a **light cone**; this entire set is invariant under Lorentz transformations. Light cones are naturally divided into future and past; the set of all points inside the future and past light cones of a point p are called **timelike separated** from p, while those outside the light cones are **spacelike separated** and those on the cones are **lightlike** or **null separated** from p. Referring back to (1.3), we see that the interval between timelike separated points is negative, between spacelike separated points is positive, and between null separated points is zero. (The interval is defined to be s^2, not the square root of this quantity.) Notice the distinction between this situation and that in the Newtonian world; here, it is impossible to say (in a coordinate-independent way) whether a point that is spacelike separated from p is in the future of p, the past of p, or "at the same time".

To probe the structure of Minkowski space in more detail, it is necessary to introduce the concepts of vectors and tensors. We will start with vectors, which should be familiar. Of course, in spacetime vectors are four-dimensional, and are often referred to as **four-vectors**. This turns out to make quite a bit of difference; for example, there is no such thing as a cross product between two four-vectors.

Beyond the simple fact of dimensionality, the most important thing to emphasize is that each vector is located at a given point in spacetime. You may be used to thinking of vectors as stretching from one point to another in space, and even of "free" vectors which you can slide carelessly from point to point. These are not useful concepts in relativity. Rather, to each point p in spacetime we associate the set of all possible vectors located at that point; this set is known as the **tangent space** at p, or T_p. The name is inspired by thinking of the set of vectors attached to a point on a simple curved two-dimensional space as comprising a plane which is tangent to the point. But inspiration aside, it is important to think of these vectors as being located at a single point, rather than stretching from one point to another. (Although this won't stop us from drawing them as arrows on spacetime diagrams.)



Later we will relate the tangent space at each point to things we can construct from the spacetime itself. For right now, just think of T_p as an abstract vector space for each point in spacetime. A **(real) vector space** is a collection of objects ("vectors") which, roughly speaking, can be added together and multiplied by real numbers in a linear way. Thus, for any two vectors V and W and real numbers a and b, we have

$$(a+b)(V+W) = aV + bV + aW + bW \ . \tag{1.22}$$

Every vector space has an origin, *i.e.* a zero vector which functions as an identity element under vector addition. In many vector spaces there are additional operations such as taking an inner (dot) product, but this is extra structure over and above the elementary concept of a vector space.

A vector is a perfectly well-defined geometric object, as is a **vector field**, defined as a set of vectors with exactly one at each point in spacetime. (The set of all the tangent spaces of a manifold M is called the **tangent bundle**, $T(M)$.) Nevertheless it is often useful for concrete purposes to decompose vectors into components with respect to some set of basis vectors. A **basis** is any set of vectors which both spans the vector space (any vector is a linear combination of basis vectors) and is linearly independent (no vector in the basis is a linear combination of other basis vectors). For any given vector space, there will be an infinite number of legitimate bases, but each basis will consist of the same number of vectors, known as the dimension of the space. (For a tangent space associated with a point in Minkowski space, the dimension is of course four.)

Let us imagine that at each tangent space we set up a basis of four vectors $\hat{e}_{(\mu)}$, with $\mu \in \{0, 1, 2, 3\}$ as usual. In fact let us say that each basis is adapted to the coordinates x^μ; that is, the basis vector $\hat{e}_{(1)}$ is what we would normally think of pointing along the x-axis, etc. It is by no means necessary that we choose a basis which is adapted to any coordinate system at all, although it is often convenient. (We really could be more precise here, but later on we will repeat the discussion at an excruciating level of precision, so some sloppiness now is forgivable.) Then any abstract vector A can be written as a linear combination of basis vectors:

$$A = A^\mu \hat{e}_{(\mu)} . \tag{1.23}$$

The coefficients A^μ are the **components** of the vector A. More often than not we will forget the basis entirely and refer somewhat loosely to "the vector A^μ", but keep in mind that this is shorthand. The real vector is an abstract geometrical entity, while the components are just the coefficients of the basis vectors in some convenient basis. (Since we will usually suppress the explicit basis vectors, the indices will usually label components of vectors and tensors. This is why there are parentheses around the indices on the basis vectors, to remind us that this is a collection of vectors, not components of a single vector.)

A standard example of a vector in spacetime is the tangent vector to a curve. A parameterized curve or path through spacetime is specified by the coordinates as a function of the parameter, *e.g.* $x^\mu(\lambda)$. The tangent vector $V(\lambda)$ has components

$$V^\mu = \frac{dx^\mu}{d\lambda} . \tag{1.24}$$

The entire vector is thus $V = V^\mu \hat{e}_{(\mu)}$. Under a Lorentz transformation the coordinates x^μ change according to (1.11), while the parameterization λ is unaltered; we can therefore deduce that the components of the tangent vector must change as

$$V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu . \tag{1.25}$$

However, the vector itself (as opposed to its components in some coordinate system) is invariant under Lorentz transformations. We can use this fact to derive the transformation properties of the basis vectors. Let us refer to the set of basis vectors in the transformed coordinate system as $\hat{e}_{(\nu')}$. Since the vector is invariant, we have

$$\begin{aligned} V = V^\mu \hat{e}_{(\mu)} &= V^{\nu'} \hat{e}_{(\nu')} \\ &= \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')} . \end{aligned} \tag{1.26}$$

But this relation must hold no matter what the numerical values of the components V^μ are. Therefore we can say

$$\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_\mu \hat{e}_{(\nu')} . \tag{1.27}$$

To get the new basis $\hat{e}_{(\nu')}$ in terms of the old one $\hat{e}_{(\mu)}$ we should multiply by the inverse of the Lorentz transformation $\Lambda^{\nu'}{}_{\mu}$. But the inverse of a Lorentz transformation from the unprimed to the primed coordinates is also a Lorentz transformation, this time from the primed to the unprimed systems. We will therefore introduce a somewhat subtle notation, by writing using the same symbol for both matrices, just with primed and unprimed indices adjusted. That is,

$$(\Lambda^{-1})^{\nu'}{}_{\mu} = \Lambda_{\nu'}{}^{\mu} , \tag{1.28}$$

or

$$\Lambda_{\nu'}{}^{\mu}\Lambda^{\sigma'}{}_{\mu} = \delta^{\sigma'}_{\nu'} , \quad \Lambda_{\nu'}{}^{\mu}\Lambda^{\nu'}{}_{\rho} = \delta^{\mu}_{\rho} , \tag{1.29}$$

where δ^{μ}_{ρ} is the traditional Kronecker delta symbol in four dimensions. (Note that Schutz uses a different convention, always arranging the two indices northwest/southeast; the important thing is where the primes go.) From (1.27) we then obtain the transformation rule for basis vectors:

$$\hat{e}_{(\nu')} = \Lambda_{\nu'}{}^{\mu}\hat{e}_{(\mu)} . \tag{1.30}$$

Therefore the set of basis vectors transforms via the inverse Lorentz transformation of the coordinates or vector components.

It is worth pausing a moment to take all this in. We introduced coordinates labeled by upper indices, which transformed in a certain way under Lorentz transformations. We then considered vector components which also were written with upper indices, which made sense since they transformed in the same way as the coordinate functions. (In a fixed coordinate system, each of the four coordinates x^μ can be thought of as a function on spacetime, as can each of the four components of a vector field.) The basis vectors associated with the coordinate system transformed via the inverse matrix, and were labeled by a lower index. This notation ensured that the invariant object constructed by summing over the components and basis vectors was left unchanged by the transformation, just as we would wish. It's probably not giving too much away to say that this will continue to be the case for more complicated objects with multiple indices (tensors).

Once we have set up a vector space, there is an associated vector space (of equal dimension) which we can immediately define, known as the **dual vector space**. The dual space is usually denoted by an asterisk, so that the dual space to the tangent space T_p is called the **cotangent space** and denoted T_p^*. The dual space is the space of all linear maps from the original vector space to the real numbers; in math lingo, if $\omega \in T_p^*$ is a dual vector, then it acts as a map such that:

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R} , \tag{1.31}$$

where V, W are vectors and a, b are real numbers. The nice thing about these maps is that they form a vector space themselves; thus, if ω and η are dual vectors, we have

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V) . \tag{1.32}$$

To make this construction somewhat more concrete, we can introduce a set of basis dual vectors $\hat{\theta}^{(\nu)}$ by demanding

$$\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^{\nu}_{\mu} \ . \tag{1.33}$$

Then every dual vector can be written in terms of its components, which we label with lower indices:

$$\omega = \omega_\mu \hat{\theta}^{(\mu)} \ . \tag{1.34}$$

In perfect analogy with vectors, we will usually simply write ω_μ to stand for the entire dual vector. In fact, you will sometime see elements of T_p (what we have called vectors) referred to as **contravariant vectors**, and elements of T_p^* (what we have called dual vectors) referred to as **covariant vectors**. Actually, if you just refer to ordinary vectors as vectors with upper indices and dual vectors as vectors with lower indices, nobody should be offended. Another name for dual vectors is **one-forms**, a somewhat mysterious designation which will become clearer soon.

The component notation leads to a simple way of writing the action of a dual vector on a vector:

$$\begin{aligned} \omega(V) &= \omega_\mu V^\nu \hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \delta^\mu_\nu \\ &= \omega_\mu V^\mu \in \mathbf{R} \ . \end{aligned} \tag{1.35}$$

This is why it is rarely necessary to write the basis vectors (and dual vectors) explicitly; the components do all of the work. The form of (1.35) also suggests that we can think of vectors as linear maps on dual vectors, by defining

$$V(\omega) \equiv \omega(V) = \omega_\mu V^\mu \ . \tag{1.36}$$

Therefore, the dual space to the dual vector space is the original vector space itself.

Of course in spacetime we will be interested not in a single vector space, but in fields of vectors and dual vectors. (The set of all cotangent spaces over M is the **cotangent bundle**, $T^*(M)$.) In that case the action of a dual vector field on a vector field is not a single number, but a **scalar** (or just "function") on spacetime. A scalar is a quantity without indices, which is unchanged under Lorentz transformations.

We can use the same arguments that we earlier used for vectors to derive the transformation properties of dual vectors. The answers are, for the components,

$$\omega_{\mu'} = \Lambda_{\mu'}{}^{\nu}\omega_\nu \ , \tag{1.37}$$

and for basis dual vectors,

$$\hat{\theta}^{(\rho')} = \Lambda^{\rho'}{}_{\sigma}\hat{\theta}^{(\sigma)} \ . \tag{1.38}$$

This is just what we would expect from index placement; the components of a dual vector transform under the inverse transformation of those of a vector. Note that this ensures that the scalar (1.35) is invariant under Lorentz transformations, just as it should be.

Let's consider some examples of dual vectors, first in other contexts and then in Minkowski space. Imagine the space of n-component column vectors, for some integer n. Then the dual space is that of n-component row vectors, and the action is ordinary matrix multiplication:

$$\begin{aligned} V &= \begin{pmatrix} V^1 \\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ V^n \end{pmatrix} , \quad \omega = (\omega_1 \; \omega_2 \; \cdots \; \omega_n) \; , \\ \omega(V) &= (\omega_1 \; \omega_2 \; \cdots \; \omega_n) \begin{pmatrix} V^1 \\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ V^n \end{pmatrix} = \omega_i V^i \; . \end{aligned} \tag{1.39}$$

Another familiar example occurs in quantum mechanics, where vectors in the Hilbert space are represented by kets, $|\psi\rangle$. In this case the dual space is the space of bras, $\langle\phi|$, and the action gives the number $\langle\phi|\psi\rangle$. (This is a complex number in quantum mechanics, but the idea is precisely the same.)

In spacetime the simplest example of a dual vector is the **gradient** of a scalar function, the set of partial derivatives with respect to the spacetime coordinates, which we denote by "d":

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x^\mu}\hat{\theta}^{(\mu)} \; . \tag{1.40}$$

The conventional chain rule used to transform partial derivatives amounts in this case to the transformation rule of components of dual vectors:

$$\begin{aligned} \frac{\partial\phi}{\partial x^{\mu'}} &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial\phi}{\partial x^\mu} \\ &= \Lambda_{\mu'}{}^{\mu}\frac{\partial\phi}{\partial x^\mu} \; , \end{aligned} \tag{1.41}$$

where we have used (1.11) and (1.28) to relate the Lorentz transformation to the coordinates. The fact that the gradient is a dual vector leads to the following shorthand notations for partial derivatives:

$$\frac{\partial\phi}{\partial x^\mu} = \partial_\mu\phi = \phi,_{\,\mu} \; . \tag{1.42}$$

(Very roughly speaking, "x^μ has an upper index, but when it is in the denominator of a derivative it implies a lower index on the resulting object.") I'm not a big fan of the comma notation, but we will use ∂_μ all the time. Note that the gradient does in fact act in a natural way on the example we gave above of a vector, the tangent vector to a curve. The result is ordinary derivative of the function along the curve:

$$\partial_\mu \phi \frac{\partial x^\mu}{\partial \lambda} = \frac{d\phi}{d\lambda} . \tag{1.43}$$

As a final note on dual vectors, there is a way to represent them as pictures which is consistent with the picture of vectors as arrows. See the discussion in Schutz, or in MTW (where it is taken to dizzying extremes).

A straightforward generalization of vectors and dual vectors is the notion of a **tensor**. Just as a dual vector is a linear map from vectors to $\mathbf{R}$, a tensor T of type (or rank) (k, l) is a multilinear map from a collection of dual vectors and vectors to $\mathbf{R}$:

$$\begin{array}{rl} T: & T_p^* \times \cdots \times T_p^* \times T_p \times \cdots \times T_p \rightarrow \mathbf{R} \\ & \quad (k \text{ times}) \qquad\quad (l \text{ times}) \end{array} \tag{1.44}$$

Here, "$\times$" denotes the Cartesian product, so that for example $T_p \times T_p$ is the space of ordered pairs of vectors. Multilinearity means that the tensor acts linearly in each of its arguments; for instance, for a tensor of type $(1, 1)$, we have

$$T(a\omega + b\eta, cV + dW) = acT(\omega, V) + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W) . \tag{1.45}$$

From this point of view, a scalar is a type $(0, 0)$ tensor, a vector is a type $(1, 0)$ tensor, and a dual vector is a type $(0, 1)$ tensor.

The space of all tensors of a fixed type (k, l) forms a vector space; they can be added together and multiplied by real numbers. To construct a basis for this space, we need to define a new operation known as the **tensor product**, denoted by $\otimes$. If T is a (k, l) tensor and S is a (m, n) tensor, we define a $(k + m, l + n)$ tensor $T \otimes S$ by

$$\begin{aligned} &T \otimes S(\omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)}) \\ &\quad = T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) S(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots, V^{(l+n)}) . \end{aligned} \tag{1.46}$$

(Note that the $\omega^{(i)}$ and $V^{(i)}$ are distinct dual vectors and vectors, not components thereof.) In other words, first act T on the appropriate set of dual vectors and vectors, and then act S on the remainder, and then multiply the answers. Note that, in general, $T \otimes S \neq S \otimes T$.

It is now straightforward to construct a basis for the space of all (k, l) tensors, by taking tensor products of basis vectors and dual vectors; this basis will consist of all tensors of the form

$$\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)} . \tag{1.47}$$

In a 4-dimensional spacetime there will be 4^{k+l} basis tensors in all. In component notation we then write our arbitrary tensor as

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)} . \tag{1.48}$$

Alternatively, we could define the components by acting the tensor on basis vectors and dual vectors:

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(\hat{\theta}^{(\mu_1)}, \ldots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \ldots, \hat{e}_{(\nu_l)}) . \tag{1.49}$$

You can check for yourself, using (1.33) and so forth, that these equations all hang together properly.

As with vectors, we will usually take the shortcut of denoting the tensor T by its components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$. The action of the tensors on a set of vectors and dual vectors follows the pattern established in (1.35):

$$T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\omega^{(1)}_{\mu_1} \cdots \omega^{(k)}_{\mu_k} V^{(1)\nu_1} \cdots V^{(l)\nu_l} . \tag{1.50}$$

The order of the indices is obviously important, since the tensor need not act in the same way on its various arguments. Finally, the transformation of tensor components under Lorentz transformations can be derived by applying what we already know about the transformation of basis vectors and dual vectors. The answer is just what you would expect from index placement,

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu'_k}{}_{\mu_k}\Lambda_{\nu'_1}{}^{\nu_1} \cdots \Lambda_{\nu'_l}{}^{\nu_l} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} . \tag{1.51}$$

Thus, each upper index gets transformed like a vector, and each lower index gets transformed like a dual vector.

Although we have defined tensors as linear maps from sets of vectors and tangent vectors to $\mathbf{R}$, there is nothing that forces us to act on a full collection of arguments. Thus, a $(1,1)$ tensor also acts as a map from vectors to vectors:

$$T^\mu{}_\nu : \ V^\nu \to T^\mu{}_\nu V^\nu . \tag{1.52}$$

You can check for yourself that $T^\mu{}_\nu V^\nu$ is a vector (*i.e.* obeys the vector transformation law). Similarly, we can act one tensor on (all or part of) another tensor to obtain a third tensor. For example,

$$U^\mu{}_\nu = T^{\mu\rho}{}_\sigma S^\sigma{}_{\rho\nu} \tag{1.53}$$

is a perfectly good $(1,1)$ tensor.

You may be concerned that this introduction to tensors has been somewhat too brief, given the esoteric nature of the material. In fact, the notion of tensors does not require a great deal of effort to master; it's just a matter of keeping the indices straight, and the rules for manipulating them are very natural. Indeed, a number of books like to *define* tensors as collections of numbers transforming according to (1.51). While this is operationally useful, it tends to obscure the deeper meaning of tensors as geometrical entities with a life independent of any chosen coordinate system. There is, however, one subtlety which we have glossed over. The notions of dual vectors and tensors and bases and linear maps belong to the realm of linear algebra, and are appropriate whenever we have an abstract vector space at hand. In the case of interest to us we have not just a vector space, but a vector space at each point in spacetime. More often than not we are interested in tensor fields, which can be thought of as tensor-valued functions on spacetime. Fortunately, none of the manipulations we defined above really care whether we are dealing with a single vector space or a collection of vector spaces, one for each event. We will be able to get away with simply calling things functions of x^μ when appropriate. However, you should keep straight the logical independence of the notions we have introduced and their specific application to spacetime and relativity.

Now let's turn to some examples of tensors. First we consider the previous example of column vectors and their duals, row vectors. In this system a $(1,1)$ tensor is simply a matrix, $M^i{}_j$. Its action on a pair (ω, V) is given by usual matrix multiplication:

$$M(\omega, V) = (\omega_1\ \omega_2\ \cdots\ \omega_n) \begin{pmatrix} M^1{}_1 & M^1{}_2 & \cdots & M^1{}_n \\ M^2{}_1 & M^2{}_2 & \cdots & M^2{}_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ M^n{}_1 & M^n{}_2 & \cdots & M^n{}_n \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ V^n \end{pmatrix} = \omega_i M^i{}_j V^j \ . \qquad (1.54)$$

If you like, feel free to think of tensors as "matrices with an arbitrary number of indices."

In spacetime, we have already seen some examples of tensors without calling them that. The most familiar example of a $(0,2)$ tensor is the metric, $\eta_{\mu\nu}$. The action of the metric on two vectors is so useful that it gets its own name, the **inner product** (or dot product):

$$\eta(V, W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W \ . \qquad (1.55)$$

Just as with the conventional Euclidean dot product, we will refer to two vectors whose dot product vanishes as **orthogonal**. Since the dot product is a scalar, it is left invariant under Lorentz transformations; therefore the basis vectors of any Cartesian inertial frame, which are chosen to be orthogonal by definition, are still orthogonal after a Lorentz transformation (despite the "scissoring together" we noticed earlier). The **norm** of a vector is defined to be inner product of the vector with itself; unlike in Euclidean space, this number is not positive definite:

$$\text{if} \quad \eta_{\mu\nu} V^\mu V^\nu \quad \text{is} \begin{cases} < 0 \ , & V^\mu \text{ is timelike} \\ = 0 \ , & V^\mu \text{ is lightlike or null} \\ > 0 \ , & V^\mu \text{ is spacelike} \ . \end{cases}$$

(A vector can have zero norm without being the zero vector.) You will notice that the terminology is the same as that which we earlier used to classify the relationship between two points in spacetime; it's no accident, of course, and we will go into more detail later.

Another tensor is the Kronecker delta δ^μ_ν, of type $(1,1)$, which you already know the components of. Related to this and the metric is the **inverse metric** $\eta^{\mu\nu}$, a type $(2,0)$ tensor defined as the inverse of the metric:

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\rho_\mu \ . \tag{1.56}$$

In fact, as you can check, the inverse metric has exactly the same components as the metric itself. (This is only true in flat space in Cartesian coordinates, and will fail to hold in more general situations.) There is also the **Levi-Civita tensor**, a $(0,4)$ tensor:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 \text{ if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 \text{ if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 \text{ otherwise .} \end{cases} \tag{1.57}$$

Here, a "permutation of 0123" is an ordering of the numbers 0, 1, 2, 3 which can be obtained by starting with 0123 and exchanging two of the digits; an even permutation is obtained by an even number of such exchanges, and an odd permutation is obtained by an odd number. Thus, for example, $\epsilon_{0321} = -1$.

It is a remarkable property of the above tensors – the metric, the inverse metric, the Kronecker delta, and the Levi-Civita tensor – that, even though they all transform according to the tensor transformation law (1.51), their components remain unchanged in *any* Cartesian coordinate system in flat spacetime. In some sense this makes them bad examples of tensors, since most tensors do not have this property. In fact, even these tensors do not have this property once we go to more general coordinate systems, with the single exception of the Kronecker delta. This tensor has exactly the same components in any coordinate system in any spacetime. This makes sense from the definition of a tensor as a linear map; the Kronecker tensor can be thought of as the identity map from vectors to vectors (or from dual vectors to dual vectors), which clearly must have the same components regardless of coordinate system. The other tensors (the metric, its inverse, and the Levi-Civita tensor) characterize the structure of spacetime, and all depend on the metric. We shall therefore have to treat them more carefully when we drop our assumption of flat spacetime.

A more typical example of a tensor is the **electromagnetic field strength tensor**. We all know that the electromagnetic fields are made up of the electric field vector E_i and the magnetic field vector B_i. (Remember that we use Latin indices for spacelike components 1,2,3.) Actually these are only "vectors" under rotations in space, not under the full Lorentz group. In fact they are components of a $(0,2)$ tensor $F_{\mu\nu}$, defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu} \ . \tag{1.58}$$

From this point of view it is easy to transform the electromagnetic fields in one reference frame to those in another, by application of (1.51). The unifying power of the tensor formalism is evident: rather than a collection of two vectors whose relationship and transformation properties are rather mysterious, we have a single tensor field to describe all of electromagnetism. (On the other hand, don't get carried away; sometimes it's more convenient to work in a single coordinate system using the electric and magnetic field vectors.)

With some examples in hand we can now be a little more systematic about some properties of tensors. First consider the operation of **contraction**, which turns a (k,l) tensor into a $(k-1,l-1)$ tensor. Contraction proceeds by summing over one upper and one lower index:

$$S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu} \ . \tag{1.59}$$

You can check that the result is a well-defined tensor. Of course it is only permissible to contract an upper index with a lower index (as opposed to two indices of the same type). Note also that the order of the indices matters, so that you can get different tensors by contracting in different ways; thus,

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu} \tag{1.60}$$

in general.

The metric and inverse metric can be used to **raise and lower indices** on tensors. That is, given a tensor $T^{\alpha\beta}{}_{\gamma\delta}$, we can use the metric to define new tensors which we choose to denote by the same letter T:

$$\begin{aligned} T^{\alpha\beta\mu}{}_{\delta} &= \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta} \ , \\ T_{\mu}{}^{\beta}{}_{\gamma\delta} &= \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta} \ , \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta} \ , \end{aligned} \tag{1.61}$$

and so forth. Notice that raising and lowering does not change the position of an index relative to other indices, and also that "free" indices (which are not summed over) must be the same on both sides of an equation, while "dummy" indices (which are summed over) only appear on one side. As an example, we can turn vectors and dual vectors into each other by raising and lowering indices:

$$\begin{aligned} V_{\mu} &= \eta_{\mu\nu} V^{\nu} \\ \omega^{\mu} &= \eta^{\mu\nu} \omega_{\nu} \ . \end{aligned} \tag{1.62}$$

This explains why the gradient in three-dimensional flat Euclidean space is usually thought of as an ordinary vector, even though we have seen that it arises as a dual vector; in Euclidean space (where the metric is diagonal with all entries +1) a dual vector is turned into a vector with precisely the same components when we raise its index. You may then wonder why we have belabored the distinction at all. One simple reason, of course, is that in a Lorentzian spacetime the components are not equal:

$$\omega^\mu = (-\omega_0, \omega_1, \omega_2, \omega_3) \ . \tag{1.63}$$

In a curved spacetime, where the form of the metric is generally more complicated, the difference is rather more dramatic. But there is a deeper reason, namely that tensors generally have a "natural" definition which is independent of the metric. Even though we will always have a metric available, it is helpful to be aware of the logical status of each mathematical object we introduce. The gradient, and its action on vectors, is perfectly well defined regardless of any metric, whereas the "gradient with upper indices" is not. (As an example, we will eventually want to take variations of functionals with respect to the metric, and will therefore have to know exactly how the functional depends on the metric, something that is easily obscured by the index notation.)

Continuing our compilation of tensor jargon, we refer to a tensor as **symmetric** in any of its indices if it is unchanged under exchange of those indices. Thus, if

$$S_{\mu\nu\rho} = S_{\nu\mu\rho} \ , \tag{1.64}$$

we say that $S_{\mu\nu\rho}$ is symmetric in its first two indices, while if

$$S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu} \ , \tag{1.65}$$

we say that $S_{\mu\nu\rho}$ is symmetric in all three of its indices. Similarly, a tensor is **antisymmetric** (or "skew-symmetric") in any of its indices if it changes sign when those indices are exchanged; thus,

$$A_{\mu\nu\rho} = -A_{\rho\nu\mu} \tag{1.66}$$

means that $A_{\mu\nu\rho}$ is antisymmetric in its first and third indices (or just "antisymmetric in μ and ρ"). If a tensor is (anti-) symmetric in all of its indices, we refer to it as simply (anti-) symmetric (sometimes with the redundant modifier "completely"). As examples, the metric $\eta_{\mu\nu}$ and the inverse metric $\eta^{\mu\nu}$ are symmetric, while the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ and the electromagnetic field strength tensor $F_{\mu\nu}$ are antisymmetric. (Check for yourself that if you raise or lower a set of indices which are symmetric or antisymmetric, they remain that way.) Notice that it makes no sense to exchange upper and lower indices with each other, so don't succumb to the temptation to think of the Kronecker delta δ^α_β as symmetric. On the other hand, the fact that lowering an index on δ^α_β gives a symmetric tensor (in fact, the metric) means that the order of indices doesn't really matter, which is why we don't keep track index placement for this one tensor.

Given any tensor, we can **symmetrize** (or antisymmetrize) any number of its upper or lower indices. To symmetrize, we take the sum of all permutations of the relevant indices and divide by the number of terms:

$$T_{(\mu_1\mu_2\cdots\mu_n)\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{sum over permutations of indices } \mu_1\cdots\mu_n\right) , \tag{1.67}$$

while antisymmetrization comes from the alternating sum:

$$T_{[\mu_1\mu_2\cdots\mu_n]\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{alternating sum over permutations of indices } \mu_1\cdots\mu_n\right) . \tag{1.68}$$

By "alternating sum" we mean that permutations which are the result of an odd number of exchanges are given a minus sign, thus:

$$T_{[\mu\nu\rho]\sigma} = \frac{1}{6}\left(T_{\mu\nu\rho\sigma} - T_{\mu\rho\nu\sigma} + T_{\rho\mu\nu\sigma} - T_{\nu\mu\rho\sigma} + T_{\nu\rho\mu\sigma} - T_{\rho\nu\mu\sigma}\right) . \tag{1.69}$$

Notice that round/square brackets denote symmetrization/antisymmetrization. Furthermore, we may sometimes want to (anti-) symmetrize indices which are not next to each other, in which case we use vertical bars to denote indices not included in the sum:

$$T_{(\mu|\nu|\rho)} = \frac{1}{2}\left(T_{\mu\nu\rho} + T_{\rho\nu\mu}\right) . \tag{1.70}$$

Finally, some people use a convention in which the factor of $1/n!$ is omitted. The one used here is a good one, since (for example) a symmetric tensor satisfies

$$S_{\mu_1\cdots\mu_n} = S_{(\mu_1\cdots\mu_n)} , \tag{1.71}$$

and likewise for antisymmetric tensors.

We have been very careful so far to distinguish clearly between things that are always true (on a manifold with arbitrary metric) and things which are only true in Minkowski space in Cartesian coordinates. One of the most important distinctions arises with **partial derivatives**. If we are working in flat spacetime with Cartesian coordinates, then the partial derivative of a (k, l) tensor is a $(k, l+1)$ tensor; that is,

$$T_{\alpha}{}^{\mu}{}_{\nu} = \partial_{\alpha} R^{\mu}{}_{\nu} \tag{1.72}$$

transforms properly under Lorentz transformations. However, this will no longer be true in more general spacetimes, and we will have to define a "covariant derivative" to take the place of the partial derivative. Nevertheless, we can still use the fact that partial derivatives give us tensor in this special case, as long as we keep our wits about us. (The one exception to this warning is the partial derivative of a scalar, $\partial_\alpha\phi$, which is a perfectly good tensor [the gradient] in any spacetime.)

We have now accumulated enough tensor know-how to illustrate some of these concepts using actual physics. Specifically, we will examine **Maxwell's equations** of electrodynamics. In 19$^{\rm th}$-century notation, these are

$$\begin{aligned}
\nabla\times\mathbf{B}-\partial_t\mathbf{E} &= 4\pi\mathbf{J}\\
\nabla\cdot\mathbf{E} &= 4\pi\rho\\
\nabla\times\mathbf{E}+\partial_t\mathbf{B} &= 0\\
\nabla\cdot\mathbf{B} &= 0\ .
\end{aligned}\tag{1.73}$$

Here, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field 3-vectors, $\mathbf{J}$ is the current, ρ is the charge density, and $\nabla\times$ and $\nabla\cdot$ are the conventional curl and divergence. These equations are invariant under Lorentz transformations, of course; that's how the whole business got started. But they don't look obviously invariant; our tensor notation can fix that. Let's begin by writing these equations in just a slightly different notation,

$$\begin{aligned}
\epsilon^{ijk}\partial_j B_k-\partial_0 E^i &= 4\pi J^i\\
\partial_i E^i &= 4\pi J^0\\
\epsilon^{ijk}\partial_j E_k+\partial_0 B^i &= 0\\
\partial_i B^i &= 0\ .
\end{aligned}\tag{1.74}$$

In these expressions, spatial indices have been raised and lowered with abandon, without any attempt to keep straight where the metric appears. This is because δ_{ij} is the metric on flat 3-space, with δ^{ij} its inverse (they are equal as matrices). We can therefore raise and lower indices at will, since the components don't change. Meanwhile, the three-dimensional Levi-Civita tensor ϵ^{ijk} is defined just as the four-dimensional one, although with one fewer index. We have replaced the charge density by J^0; this is legitimate because the density and current together form the **current 4-vector**, $J^\mu=(\rho,J^1,J^2,J^3)$.

From these expressions, and the definition (1.58) of the field strength tensor $F_{\mu\nu}$, it is easy to get a completely tensorial 20$^{\rm th}$-century version of Maxwell's equations. Begin by noting that we can express the field strength with upper indices as

$$\begin{aligned}
F^{0i} &= E^i\\
F^{ij} &= \epsilon^{ijk}B_k\ .
\end{aligned}\tag{1.75}$$

(To check this, note for example that $F^{01}=\eta^{00}\eta^{11}F_{01}$ and $F^{12}=\epsilon^{123}B_3$.) Then the first two equations in (1.74) become

$$\partial_j F^{ij}-\partial_0 F^{0i} = 4\pi J^i$$

$$\partial_i F^{0i} = 4\pi J^0 \ . \tag{1.76}$$

Using the antisymmetry of $F^{\mu\nu}$, we see that these may be combined into the single tensor equation

$$\partial_\mu F^{\nu\mu} = 4\pi J^\nu \ . \tag{1.77}$$

A similar line of reasoning, which is left as an exercise to you, reveals that the third and fourth equations in (1.74) can be written

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \ . \tag{1.78}$$

The four traditional Maxwell equations are thus replaced by two, thus demonstrating the economy of tensor notation. More importantly, however, both sides of equations (1.77) and (1.78) manifestly transform as tensors; therefore, if they are true in one inertial frame, they must be true in any Lorentz-transformed frame. This is why tensors are so useful in relativity — we often want to express relationships without recourse to any reference frame, and it is necessary that the quantities on each side of an equation transform in the same way under change of coordinates. As a matter of jargon, we will sometimes refer to quantities which are written in terms of tensors as **covariant** (which has nothing to do with "covariant" as opposed to "contravariant"). Thus, we say that (1.77) and (1.78) together serve as the covariant form of Maxwell's equations, while (1.73) or (1.74) are non-covariant.

Let us now introduce a special class of tensors, known as **differential forms** (or just "forms"). A differential p-form is a $(0,p)$ tensor which is completely antisymmetric. Thus, scalars are automatically 0-forms, and dual vectors are automatically one-forms (thus explaining this terminology from a while back). We also have the 2-form $F_{\mu\nu}$ and the 4-form $\epsilon_{\mu\nu\rho\sigma}$. The space of all p-forms is denoted Λ^p, and the space of all p-form fields over a manifold M is denoted $\Lambda^p(M)$. A semi-straightforward exercise in combinatorics reveals that the number of linearly independent p-forms on an n-dimensional vector space is $n!/(p!(n-p)!)$. So at a point on a 4-dimensional spacetime there is one linearly independent 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. There are no p-forms for $p > n$, since all of the components will automatically be zero by antisymmetry.

Why should we care about differential forms? This is a hard question to answer without some more work, but the basic idea is that forms can be both differentiated and integrated, without the help of any additional geometric structure. We will delay integration theory until later, but see how to differentiate forms shortly.

Given a p-form A and a q-form B, we can form a $(p+q)$-form known as the **wedge product** $A \wedge B$ by taking the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]} \ . \tag{1.79}$$

Thus, for example, the wedge product of two 1-forms is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu \ . \tag{1.80}$$

Note that

$$A \wedge B = (-1)^{pq} B \wedge A \ , \tag{1.81}$$

so you can alter the order of a wedge product if you are careful with signs.

The **exterior derivative** "d" allows us to differentiate p-form fields to obtain $(p+1)$-form fields. It is defined as an appropriately normalized antisymmetric partial derivative:

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]} \ . \tag{1.82}$$

The simplest example is the gradient, which is the exterior derivative of a 1-form:

$$(\mathrm{d}\phi)_\mu = \partial_\mu \phi \ . \tag{1.83}$$

The reason why the exterior derivative deserves special attention is that *it is a tensor*, even in curved spacetimes, unlike its cousin the partial derivative. Since we haven't studied curved spaces yet, we cannot prove this, but (1.82) defines an honest tensor no matter what the metric and coordinates are.

Another interesting fact about exterior differentiation is that, for any form A,

$$\mathrm{d}(\mathrm{d}A) = 0 \ , \tag{1.84}$$

which is often written $\mathrm{d}^2 = 0$. This identity is a consequence of the definition of d and the fact that partial derivatives commute, $\partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha$ (acting on anything). This leads us to the following mathematical aside, just for fun. We define a p-form A to be **closed** if $\mathrm{d}A = 0$, and **exact** if $A = \mathrm{d}B$ for some $(p-1)$-form B. Obviously, all exact forms are closed, but the converse is not necessarily true. On a manifold M, closed p-forms comprise a vector space $Z^p(M)$, and exact forms comprise a vector space $B^p(M)$. Define a new vector space as the closed forms modulo the exact forms:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)} \ . \tag{1.85}$$

This is known as the pth de Rham cohomology vector space, and depends only on the topology of the manifold M. (Minkowski space is topologically equivalent to $\mathbf{R}^4$, which is uninteresting, so that all of the $H^p(M)$ vanish for $p > 0$; for $p = 0$ we have $H^0(M) = \mathbf{R}$. Therefore in Minkowski space all closed forms are exact except for zero-forms; zero-forms can't be exact since there are no -1-forms for them to be the exterior derivative of.) It is striking that information about the topology can be extracted in this way, which essentially involves the solutions to differential equations. The dimension b_p of the space $H^p(M)$ is called the pth Betti number of M, and the Euler characteristic is given by the alternating sum

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p \ . \tag{1.86}$$

Cohomology theory is the basis for much of modern differential topology.

Moving back to reality, the final operation on differential forms we will introduce is **Hodge duality**. We define the "Hodge star operator" on an n-dimensional manifold as a map from p-forms to $(n-p)$-forms,

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p} \ , \tag{1.87}$$

mapping A to "A dual". Unlike our other operations on forms, the Hodge dual does depend on the metric of the manifold (which should be obvious, since we had to raise some indices on the Levi-Civita tensor in order to define (1.87)). Applying the Hodge star twice returns either plus or minus the original form:

$$**A = (-1)^{s+p(n-p)} A \ , \tag{1.88}$$

where s is the number of minus signs in the eigenvalues of the metric (for Minkowski space, $s = 1$).

Two facts on the Hodge dual: First, "duality" in the sense of Hodge is different than the relationship between vectors and dual vectors, although both can be thought of as the space of linear maps from the original space to $\mathbf{R}$. Notice that the dimensionality of the space of $(n-p)$-forms is equal to that of the space of p-forms, so this has at least a chance of being true. In the case of forms, the linear map defined by an $(n-p)$-form acting on a p-form is given by the dual of the wedge product of the two forms. Thus, if $A^{(n-p)}$ is an $(n-p)$-form and $B^{(p)}$ is a p-form at some point in spacetime, we have

$$*\,(A^{(n-p)} \wedge B^{(p)}) \in \mathbf{R} \ . \tag{1.89}$$

The second fact concerns differential forms in 3-dimensional Euclidean space. The Hodge dual of the wedge product of two 1-forms gives another 1-form:

$$*\,(U \wedge V)_i = \epsilon_i{}^{jk} U_j V_k \ . \tag{1.90}$$

(All of the prefactors cancel.) Since 1-forms in Euclidean space are just like vectors, we have a map from two vectors to a single vector. You should convince yourself that this is just the conventional cross product, and that the appearance of the Levi-Civita tensor explains why the cross product changes sign under parity (interchange of two coordinates, or equivalently basis vectors). This is why the cross product only exists in three dimensions — because only in three dimensions do we have an interesting map from two dual vectors to a third dual vector. If you wanted to you could define a map from $n-1$ one-forms to a single one-form, but I'm not sure it would be of any use.

Electrodynamics provides an especially compelling example of the use of differential forms. From the definition of the exterior derivative, it is clear that equation (1.78) can be concisely expressed as closure of the two-form $F_{\mu\nu}$:

$$\mathrm{d}F = 0 \ . \tag{1.91}$$

Does this mean that F is also exact? Yes; as we've noted, Minkowski space is topologically trivial, so all closed forms are exact. There must therefore be a one-form A_μ such that

$$F = \mathrm{d}A \ . \tag{1.92}$$

This one-form is the familiar **vector potential** of electromagnetism, with the 0 component given by the scalar potential, $A_0 = \phi$. If one starts from the view that the A_μ is the fundamental field of electromagnetism, then (1.91) follows as an identity (as opposed to a dynamical law, an equation of motion). Gauge invariance is expressed by the observation that the theory is invariant under $A \to A + \mathrm{d}\lambda$ for some scalar (zero-form) λ, and this is also immediate from the relation (1.92). The other one of Maxwell's equations, (1.77), can be expressed as an equation between three-forms:

$$\mathrm{d}(*F) = 4\pi(*J) \ , \tag{1.93}$$

where the current one-form J is just the current four-vector with index lowered. Filling in the details is left for you to do.

As an intriguing aside, Hodge duality is the basis for one of the hottest topics in theoretical physics today. It's hard not to notice that the equations (1.91) and (1.93) look very similar. Indeed, if we set $J_\mu = 0$, the equations are invariant under the "duality transformations"

$$\begin{aligned} F &\to *F \ , \\ *F &\to -F \ . \end{aligned} \tag{1.94}$$

We therefore say that the vacuum Maxwell's equations are duality invariant, while the invariance is spoiled in the presence of charges. We might imagine that magnetic as well as electric monopoles existed in nature; then we could add a magnetic current term $4\pi(*J_M)$ to the right hand side of (1.91), and the equations would be invariant under duality transformations plus the additional replacement $J \leftrightarrow J_M$. (Of course a nonzero right hand side to (1.91) is inconsistent with $F = \mathrm{d}A$, so this idea only works if A_μ is not a fundamental variable.) Long ago Dirac considered the idea of magnetic monopoles and showed that a necessary condition for their existence is that the fundamental monopole charge be inversely proportional to the fundamental electric charge. Now, the fundamental electric charge is a small number; electrodynamics is "weakly coupled", which is why perturbation theory is so remarkably successful in quantum electrodynamics (QED). But Dirac's condition on magnetic charges implies that a duality transformation takes a theory of weakly coupled electric charges to a theory of strongly coupled magnetic monopoles (and vice-versa). Unfortunately monopoles don't exist (as far as we know), so these ideas aren't directly applicable to electromagnetism; but there are some theories (such as supersymmetric non-abelian gauge theories) for which it has been long conjectured that some sort of duality symmetry may exist. If it did, we would have the opportunity to analyze a theory which looked strongly coupled (and therefore hard to solve) by looking at the weakly coupled dual version. Recently work by Seiberg and Witten and others has provided very strong evidence that this is exactly what happens in certain theories. The hope is that these techniques will allow us to explore various phenomena which we know exist in strongly coupled quantum field theories, such as confinement of quarks in hadrons.

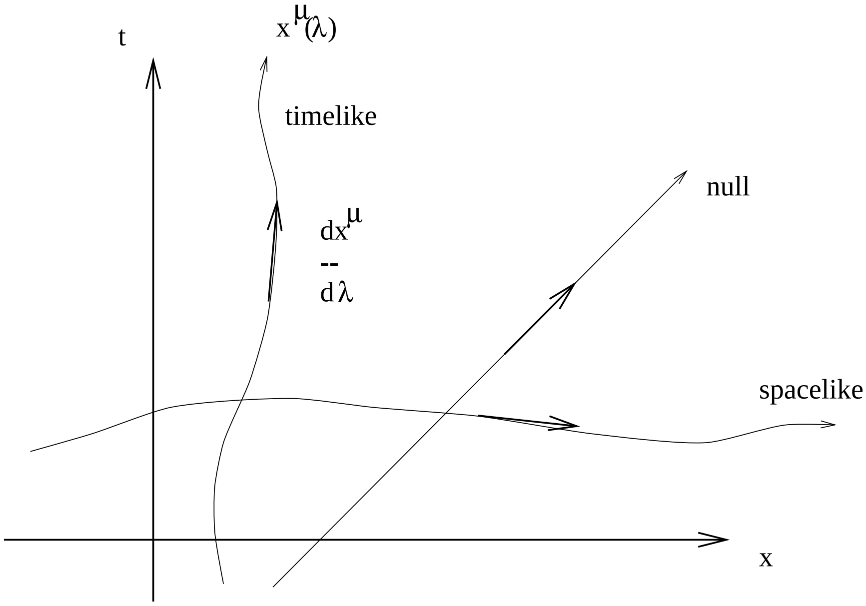
We've now gone over essentially everything there is to know about the care and feeding of tensors. In the next section we will look more carefully at the rigorous definitions of manifolds and tensors, but the basic mechanics have been pretty well covered. Before jumping to more abstract mathematics, let's review how physics works in Minkowski spacetime.

Start with the worldline of a single particle. This is specified by a map $\mathbf{R} \to M$, where M is the manifold representing spacetime; we usually think of the path as a parameterized curve $x^\mu(\lambda)$. As mentioned earlier, the tangent vector to this path is $dx^\mu/d\lambda$ (note that it depends on the parameterization). An object of primary interest is the norm of the tangent vector, which serves to characterize the path; if the tangent vector is timelike/null/spacelike at some parameter value λ, we say that the path is timelike/null/spacelike at that point. This explains why the same words are used to classify vectors in the tangent space and intervals between two points — because a straight line connecting, say, two timelike separated points will itself be timelike at every point along the path.

Nevertheless, it's important to be aware of the sleight of hand which is being pulled here. The metric, as a $(0,2)$ tensor, is a machine which acts on two vectors (or two copies of the same vector) to produce a number. It is therefore very natural to classify tangent vectors according to the sign of their norm. But the interval between two points isn't something quite so natural; it depends on a specific choice of path (a "straight line") which connects the points, and this choice in turn depends on the fact that spacetime is flat (which allows a unique choice of straight line between the points). A more natural object is the **line element**, or infinitesimal interval:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.95}$$

From this definition it is tempting to take the square root and integrate along a path to obtain a finite interval. But since ds^2 need not be positive, we define different procedures



for different cases. For spacelike paths we define the **path length**

$$\Delta s = \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda\ , \tag{1.96}$$

where the integral is taken over the path. For null paths the interval is zero, so no extra formula is required. For timelike paths we define the **proper time**

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda\ , \tag{1.97}$$

which will be positive. Of course we may consider paths that are timelike in some places and spacelike in others, but fortunately it is seldom necessary since the paths of physical particles never change their character (massive particles move on timelike paths, massless particles move on null paths). Furthermore, the phrase "proper time" is especially appropriate, since *τ actually measures the time elapsed on a physical clock carried along the path.* This point of view makes the "twin paradox" and similar puzzles very clear; two worldlines, not necessarily straight, which intersect at two different events in spacetime will have proper times measured by the integral (1.97) along the appropriate paths, and these two numbers will in general be different even if the people travelling along them were born at the same time.

Let's move from the consideration of paths in general to the paths of massive particles (which will always be timelike). Since the proper time is measured by a clock travelling on a timelike worldline, it is convenient to use τ as the parameter along the path. That is, we use (1.97) to compute $\tau(\lambda)$, which (if λ is a good parameter in the first place) we can invert to obtain $\lambda(\tau)$, after which we can think of the path as $x^\mu(\tau)$. The tangent vector in this parameterization is known as the **four-velocity**, U^μ:

$$U^\mu = \frac{dx^\mu}{d\tau} \ . \tag{1.98}$$

Since $d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$, the four-velocity is automatically normalized:

$$\eta_{\mu\nu} U^\mu U^\nu = -1 \ . \tag{1.99}$$

(It will always be negative, since we are only defining it for timelike trajectories. You could define an analogous vector for spacelike paths as well; null paths give some extra problems since the norm is zero.) In the rest frame of a particle, its four-velocity has components $U^\mu = (1, 0, 0, 0)$.

A related vector is the **energy-momentum four-vector**, defined by

$$p^\mu = mU^\mu \ , \tag{1.100}$$

where m is the mass of the particle. The mass is a fixed quantity independent of inertial frame; what you may be used to thinking of as the "rest mass." It turns out to be much more convenient to take this as the mass once and for all, rather than thinking of mass as depending on velocity. The **energy** of a particle is simply p^0, the timelike component of its energy-momentum vector. Since it's only one component of a four-vector, it is not invariant under Lorentz transformations; that's to be expected, however, since the energy of a particle at rest is not the same as that of the same particle in motion. In the particle's rest frame we have $p^0 = m$; recalling that we have set $c = 1$, we find that we have found the equation that made Einstein a celebrity, $E = mc^2$. (The field equations of general relativity are actually much more important than this one, but "$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$" doesn't elicit the visceral reaction that you get from "$E = mc^2$".) In a moving frame we can find the components of p^μ by performing a Lorentz transformation; for a particle moving with (three-) velocity v along the x axis we have

$$p^\mu = (\gamma m, v\gamma m, 0, 0) \ , \tag{1.101}$$

where $\gamma = 1/\sqrt{1 - v^2}$. For small v, this gives $p^0 = m + \frac{1}{2} m v^2$ (what we usually think of as rest energy plus kinetic energy) and $p^1 = mv$ (what we usually think of as [Newtonian] momentum). So the energy-momentum vector lives up to its name.

The centerpiece of pre-relativity physics is Newton's 2nd Law, or $\mathbf{f} = m\mathbf{a} = d\mathbf{p}/dt$. An analogous equation should hold in SR, and the requirement that it be tensorial leads us directly to introduce a force four-vector f^μ satisfying

$$f^\mu = m \frac{d^2}{d\tau^2} x^\mu(\tau) = \frac{d}{d\tau} p^\mu(\tau) \ . \tag{1.102}$$

The simplest example of a force in Newtonian physics is the force due to gravity. In relativity, however, gravity is not described by a force, but rather by the curvature of spacetime itself.

Instead, let us consider electromagnetism. The three-dimensional Lorentz force is given by $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where q is the charge on the particle. We would like a tensorial generalization of this equation. There turns out to be a unique answer:

$$f^\mu = qU^\lambda F_\lambda{}^\mu \ . \tag{1.103}$$

You can check for yourself that this reduces to the Newtonian version in the limit of small velocities. Notice how the requirement that the equation be tensorial, which is one way of guaranteeing Lorentz invariance, severely restricted the possible expressions we could get. This is an example of a very general phenomenon, in which a small number of an apparently endless variety of possible physical laws are picked out by the demands of symmetry.

Although p^μ provides a complete description of the energy and momentum of a particle, for extended systems it is necessary to go further and define the **energy-momentum tensor** (sometimes called the stress-energy tensor), $T^{\mu\nu}$. This is a symmetric $(2,0)$ tensor which tells us all we need to know about the energy-like aspects of a system: energy density, pressure, stress, and so forth. A general definition of $T^{\mu\nu}$ is "the flux of four-momentum p^μ across a surface of constant x^ν". To make this more concrete, let's consider the very general category of matter which may be characterized as a **fluid** — a continuum of matter described by macroscopic quantities such as temperature, pressure, entropy, viscosity, etc. In fact this definition is so general that it is of little use. In general relativity essentially all interesting types of matter can be thought of as **perfect fluids**, from stars to electromagnetic fields to the entire universe. Schutz defines a perfect fluid to be one with no heat conduction and no viscosity, while Weinberg defines it as a fluid which looks isotropic in its rest frame; these two viewpoints turn out to be equivalent. Operationally, you should think of a perfect fluid as one which may be completely characterized by its pressure and density.

To understand perfect fluids, let's start with the even simpler example of **dust**. Dust is defined as a collection of particles at rest with respect to each other, or alternatively as a perfect fluid with zero pressure. Since the particles all have an equal velocity in any fixed inertial frame, we can imagine a "four-velocity field" $U^\mu(x)$ defined all over spacetime. (Indeed, its components are the same at each point.) Define the **number-flux four-vector** to be

$$N^\mu = nU^\mu \ , \tag{1.104}$$

where n is the number density of the particles as measured in their rest frame. Then N^0 is the number density of particles as measured in any other frame, while N^i is the flux of particles in the x^i direction. Let's now imagine that each of the particles have the same mass m. Then in the rest frame the energy density of the dust is given by

$$\rho = nm \ . \tag{1.105}$$

By definition, the energy density completely specifies the dust. But ρ only measures the energy density in the rest frame; what about other frames? We notice that both n and m are 0-components of four-vectors in their rest frame; specifically, $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$. Therefore ρ is the $\mu = 0$, $\nu = 0$ component of the tensor $p \otimes N$ as measured in its rest frame. We are therefore led to define the energy-momentum tensor for dust:

$$T^{\mu\nu}_{\rm dust} = p^\mu N^\nu = nmU^\mu U^\nu = \rho U^\mu U^\nu \ , \tag{1.106}$$

where ρ is defined as the energy density in the rest frame.

Having mastered dust, more general perfect fluids are not much more complicated. Remember that "perfect" can be taken to mean "isotropic in its rest frame." This in turn means that $T^{\mu\nu}$ is diagonal — there is no net flux of any component of momentum in an orthogonal direction. Furthermore, the nonzero spacelike components must all be equal, $T^{11} = T^{22} = T^{33}$. The only two independent numbers are therefore T^{00} and one of the T^{ii}; we can choose to call the first of these the energy density ρ, and the second the pressure p. (Sorry that it's the same letter as the momentum.) The energy-momentum tensor of a perfect fluid therefore takes the following form in its rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \ . \tag{1.107}$$

We would like, of course, a formula which is good in any frame. For dust we had $T^{\mu\nu} = \rho U^\mu U^\nu$, so we might begin by guessing $(\rho + p)U^\mu U^\nu$, which gives

$$\begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{1.108}$$

To get the answer we want we must therefore add

$$\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \ . \tag{1.109}$$

Fortunately, this has an obvious covariant generalization, namely $p\eta^{\mu\nu}$. Thus, the general form of the energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu} \ . \tag{1.110}$$

This is an important formula for applications such as stellar structure and cosmology.

As further examples, let's consider the energy-momentum tensors of electromagnetism and scalar field theory. Without any explanation at all, these are given by

$$T^{\mu\nu}_{\rm e+m} = \frac{-1}{4\pi}(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}) \ , \tag{1.111}$$

and

$$T^{\mu\nu}_{\rm scalar} = \eta^{\mu\lambda}\eta^{\nu\sigma}\partial_\lambda\phi\partial_\sigma\phi - \frac{1}{2}\eta^{\mu\nu}(\eta^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi + m^2\phi^2) \ . \tag{1.112}$$

You can check for yourself that, for example, T^{00} in each case is equal to what you would expect the energy density to be.

Besides being symmetric, $T^{\mu\nu}$ has the even more important property of being *conserved.* In this context, conservation is expressed as the vanishing of the "divergence":

$$\partial_\mu T^{\mu\nu} = 0 \ . \tag{1.113}$$

This is a set of four equations, one for each value of ν. The $\nu = 0$ equation corresponds to conservation of energy, while $\partial_\mu T^{\mu k} = 0$ expresses conservation of the $k^{\rm th}$ component of the momentum. We are not going to prove this in general; the proof follows for any individual source of matter from the equations of motion obeyed by that kind of matter. In fact, one way to define $T^{\mu\nu}$ would be "a $(2,0)$ tensor with units of energy per volume, which is conserved." You can prove conservation of the energy-momentum tensor for electromagnetism, for example, by taking the divergence of (1.111) and using Maxwell's equations as previously discussed.

A final aside: we have already mentioned that in general relativity gravitation does not count as a "force." As a related point, the gravitational field also does not have an energy-momentum tensor. In fact it is very hard to come up with a sensible local expression for the energy of a gravitational field; a number of suggestions have been made, but they all have their drawbacks. Although there is no "correct" answer, it is an important issue from the point of view of asking seemingly reasonable questions such as "What is the energy emitted per second from a binary pulsar as the result of gravitational radiation?"

2 Manifolds

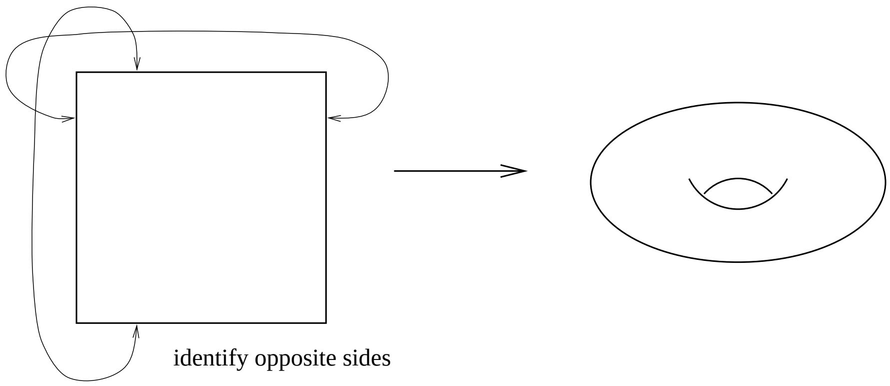
After the invention of special relativity, Einstein tried for a number of years to invent a Lorentz-invariant theory of gravity, without success. His eventual breakthrough was to replace Minkowski spacetime with a curved spacetime, where the curvature was created by (and reacted back on) energy and momentum. Before we explore how this happens, we have to learn a bit about the mathematics of curved spaces. First we will take a look at manifolds in general, and then in the next section study curvature. In the interest of generality we will usually work in n dimensions, although you are permitted to take $n = 4$ if you like.

A manifold (or sometimes "differentiable manifold") is one of the most fundamental concepts in mathematics and physics. We are all aware of the properties of n-dimensional Euclidean space, $\mathbf{R}^n$, the set of n-tuples $(x^1, \ldots, x^n)$. The notion of a manifold captures the idea of a space which may be curved and have a complicated topology, but in local regions looks just like $\mathbf{R}^n$. (Here by "looks like" we do not mean that the metric is the same, but only basic notions of analysis like open sets, functions, and coordinates.) The entire manifold is constructed by smoothly sewing together these local regions. Examples of manifolds include:

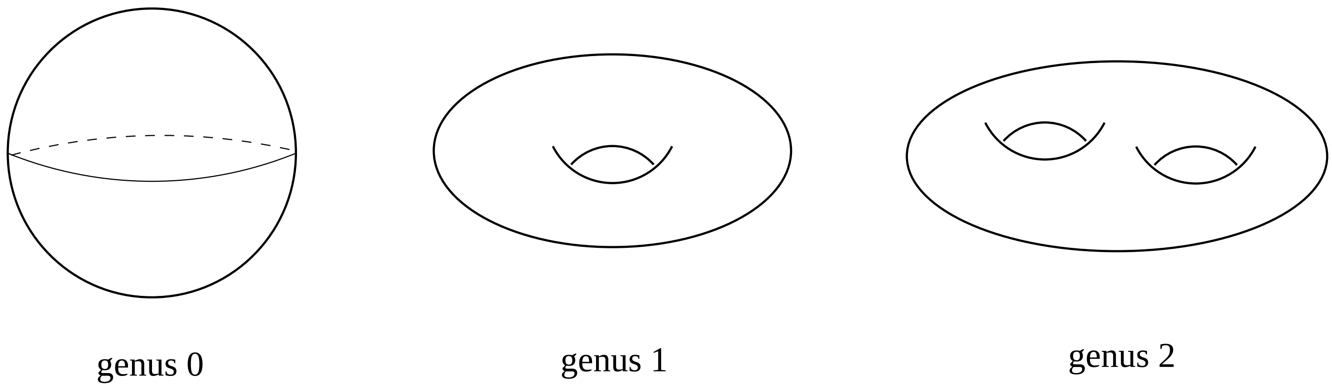
- $\mathbf{R}^n$ itself, including the line ($\mathbf{R}$), the plane ($\mathbf{R}^2$), and so on. This should be obvious, since $\mathbf{R}^n$ looks like $\mathbf{R}^n$ not only locally but globally.

- The n-sphere, S^n. This can be defined as the locus of all points some fixed distance from the origin in $\mathbf{R}^{n+1}$. The circle is of course S^1, and the two-sphere S^2 will be one of our favorite examples of a manifold.

- The n-torus T^n results from taking an n-dimensional cube and identifying opposite sides. Thus T^2 is the traditional surface of a doughnut.



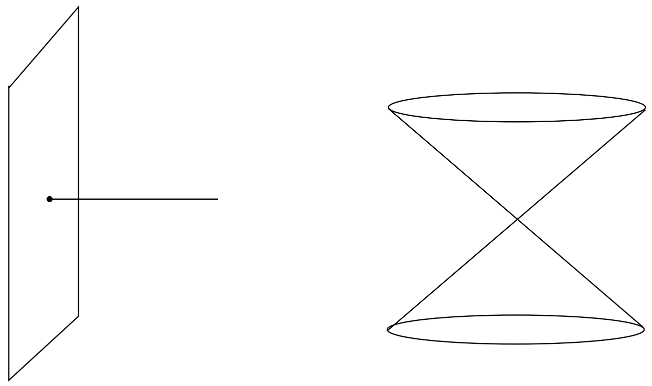
- A Riemann surface of genus g is essentially a two-torus with g holes instead of just one. S^2 may be thought of as a Riemann surface of genus zero. For those of you who know what the words mean, every "compact orientable boundaryless" two-dimensional manifold is a Riemann surface of some genus.



- More abstractly, a set of continuous transformations such as rotations in $\mathbf{R}^n$ forms a manifold. Lie groups are manifolds which also have a group structure.

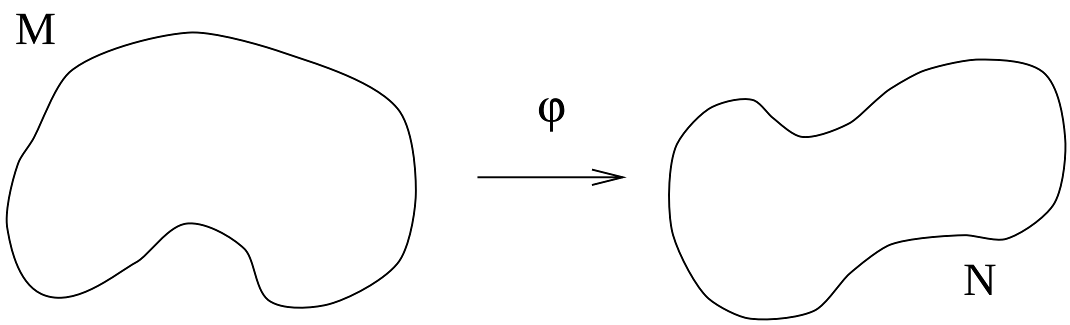
- The direct product of two manifolds is a manifold. That is, given manifolds M and M' of dimension n and n', we can construct a manifold $M \times M'$, of dimension $n + n'$, consisting of ordered pairs (p, p') for all $p \in M$ and $p' \in M'$.

With all of these examples, the notion of a manifold may seem vacuous; what isn't a manifold? There are plenty of things which are not manifolds, because somewhere they do not look locally like $\mathbf{R}^n$. Examples include a one-dimensional line running into a two-dimensional plane, and two cones stuck together at their vertices. (A single cone is okay; you can imagine smoothing out the vertex.)

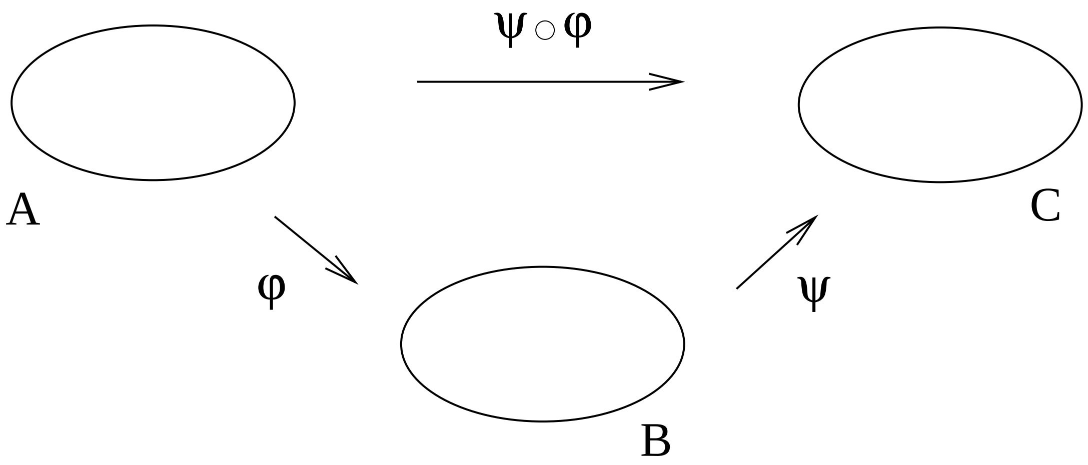


We will now approach the rigorous definition of this simple idea, which requires a number of preliminary definitions. Many of them are pretty clear anyway, but it's nice to be complete.

The most elementary notion is that of a **map** between two sets. (We assume you know what a set is.) Given two sets M and N, a map $\phi : M \to N$ is a relationship which assigns, to each element of M, exactly one element of N. A map is therefore just a simple generalization of a function. The canonical picture of a map looks like this:

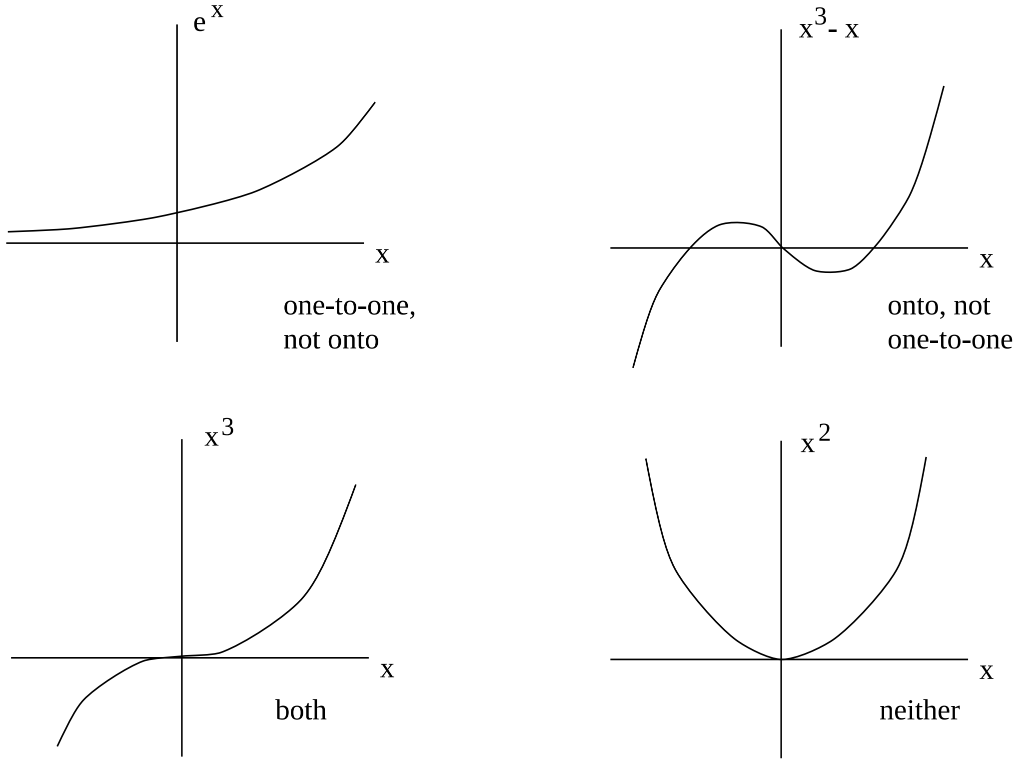


Given two maps $\phi : A \to B$ and $\psi : B \to C$, we define the **composition** $\psi \circ \phi : A \to C$ by the operation $(\psi \circ \phi)(a) = \psi(\phi(a))$. So $a \in A$, $\phi(a) \in B$, and thus $(\psi \circ \phi)(a) \in C$. The order in which the maps are written makes sense, since the one on the right acts first. In pictures:

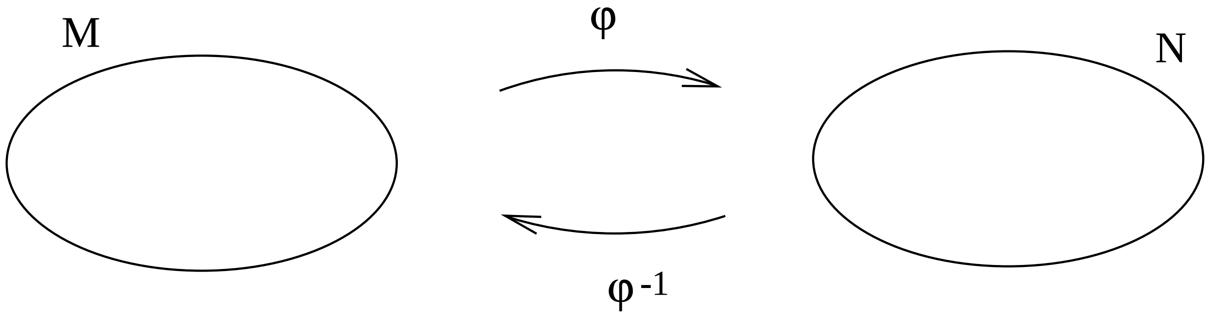


A map ϕ is called **one-to-one** (or "injective") if each element of N has at most one element of M mapped into it, and **onto** (or "surjective") if each element of N has at least one element of M mapped into it. (If you think about it, a better name for "one-to-one" would be "two-to-two".) Consider a function $\phi : \mathbf{R} \to \mathbf{R}$. Then $\phi(x) = e^x$ is one-to-one, but not onto; $\phi(x) = x^3 - x$ is onto, but not one-to-one; $\phi(x) = x^3$ is both; and $\phi(x) = x^2$ is neither.

The set M is known as the **domain** of the map ϕ, and the set of points in N which M gets mapped into is called the **image** of ϕ. For some subset $U \subset N$, the set of elements of M which get mapped to U is called the **preimage** of U under ϕ, or $\phi^{-1}(U)$. A map which is



both one-to-one and onto is known as **invertible** (or "bijective"). In this case we can define the **inverse map** $\phi^{-1} : N \to M$ by $(\phi^{-1} \circ \phi)(a) = a$. (Note that the same symbol ϕ^{-1} is used for both the preimage and the inverse map, even though the former is always defined and the latter is only defined in some special cases.) Thus:



The notion of **continuity** of a map between topological spaces (and thus manifolds) is actually a very subtle one, the precise formulation of which we won't really need. However the intuitive notions of continuity and differentiability of maps $\phi : \mathbf{R}^m \to \mathbf{R}^n$ between Euclidean spaces are useful. A map from $\mathbf{R}^m$ to $\mathbf{R}^n$ takes an m-tuple $(x^1, x^2, \ldots, x^m)$ to an n-tuple $(y^1, y^2, \ldots, y^n)$, and can therefore be thought of as a collection of n functions ϕ^i of

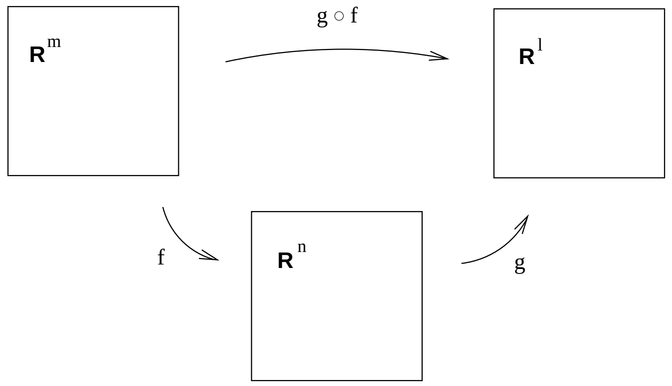
m variables:

$$\begin{aligned} y^1 &= \phi^1(x^1, x^2, \ldots, x^m) \\ y^2 &= \phi^2(x^1, x^2, \ldots, x^m) \\ &\ \ \vdots \\ y^n &= \phi^n(x^1, x^2, \ldots, x^m)\ . \end{aligned} \tag{2.1}$$

We will refer to any one of these functions as C^p if it is continuous and p-times differentiable, and refer to the entire map $\phi : \mathbf{R}^m \to \mathbf{R}^n$ as C^p if each of its component functions are at least C^p. Thus a C^0 map is continuous but not necessarily differentiable, while a C^∞ map is continuous and can be differentiated as many times as you like. C^∞ maps are sometimes called **smooth**. We will call two sets M and N **diffeomorphic** if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$; the map ϕ is then called a diffeomorphism.

Aside: The notion of two spaces being diffeomorphic only applies to manifolds, where a notion of differentiability is inherited from the fact that the space resembles $\mathbf{R}^n$ locally. But "continuity" of maps between topological spaces (not necessarily manifolds) can be defined, and we say that two such spaces are "homeomorphic," which means "topologically equivalent to," if there is a continuous map between them with a continuous inverse. It is therefore conceivable that spaces exist which are homeomorphic but not diffeomorphic; topologically the same but with distinct "differentiable structures." In 1964 Milnor showed that S^7 had 28 different differentiable structures; it turns out that for $n < 7$ there is only one differentiable structure on S^n, while for $n > 7$ the number grows very large. $\mathbf{R}^4$ has infinitely many differentiable structures.

One piece of conventional calculus that we will need later is the **chain rule**. Let us imagine that we have maps $f : \mathbf{R}^m \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^l$, and therefore the composition $(g \circ f) : \mathbf{R}^m \to \mathbf{R}^l$.



We can represent each space in terms of coordinates: x^a on $\mathbf{R}^m$, y^b on $\mathbf{R}^n$, and z^c on $\mathbf{R}^l$, where the indices range over the appropriate values. The chain rule relates the partial derivatives of the composition to the partial derivatives of the individual maps:

$$\frac{\partial}{\partial x^a}(g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a}\frac{\partial g^c}{\partial y^b} \ . \tag{2.2}$$

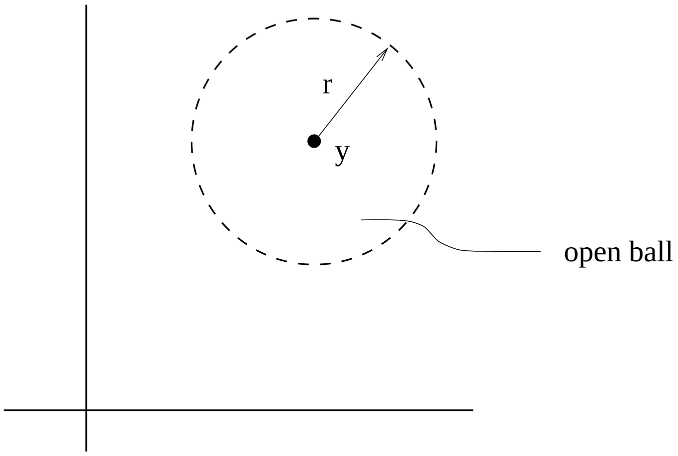
This is usually abbreviated to

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial y^b} \ . \tag{2.3}$$

There is nothing illegal or immoral about using this form of the chain rule, but you should be able to visualize the maps that underlie the construction. Recall that when $m = n$ the determinant of the matrix $\partial y^b / \partial x^a$ is called the **Jacobian** of the map, and the map is invertible whenever the Jacobian is nonzero.

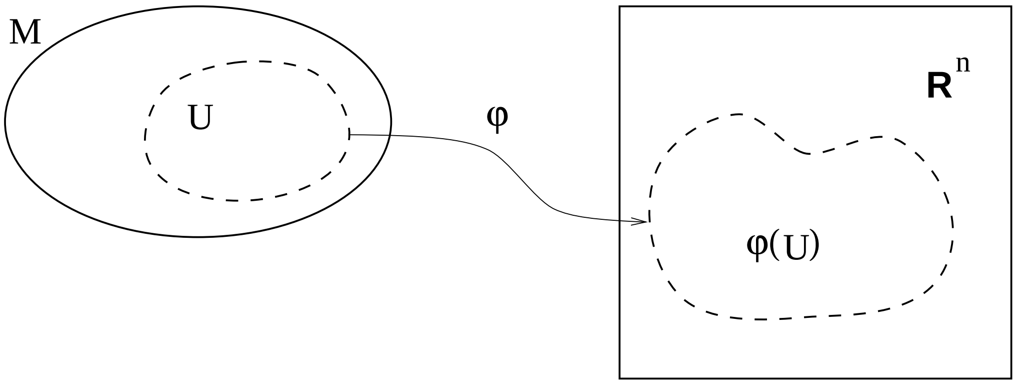
These basic definitions were presumably familiar to you, even if only vaguely remembered. We will now put them to use in the rigorous definition of a manifold. Unfortunately, a somewhat baroque procedure is required to formalize this relatively intuitive notion. We will first have to define the notion of an open set, on which we can put coordinate systems, and then sew the open sets together in an appropriate way.

Start with the notion of an **open ball**, which is the set of all points x in $\mathbf{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbf{R}^n$ and $r \in \mathbf{R}$, where $|x - y| = [\sum_i (x^i - y^i)^2]^{1/2}$. Note that this is a strict inequality — the open ball is the interior of an n-sphere of radius r centered at y.



An **open set** in $\mathbf{R}^n$ is a set constructed from an arbitrary (maybe infinite) union of open balls. In other words, $V \subset \mathbf{R}^n$ is open if, for any $y \in V$, there is an open ball centered at y which is completely inside V. Roughly speaking, an open set is the interior of some $(n-1)$-dimensional closed surface (or the union of several such interiors). By defining a notion of open sets, we have equipped $\mathbf{R}^n$ with a topology — in this case, the "standard metric topology."

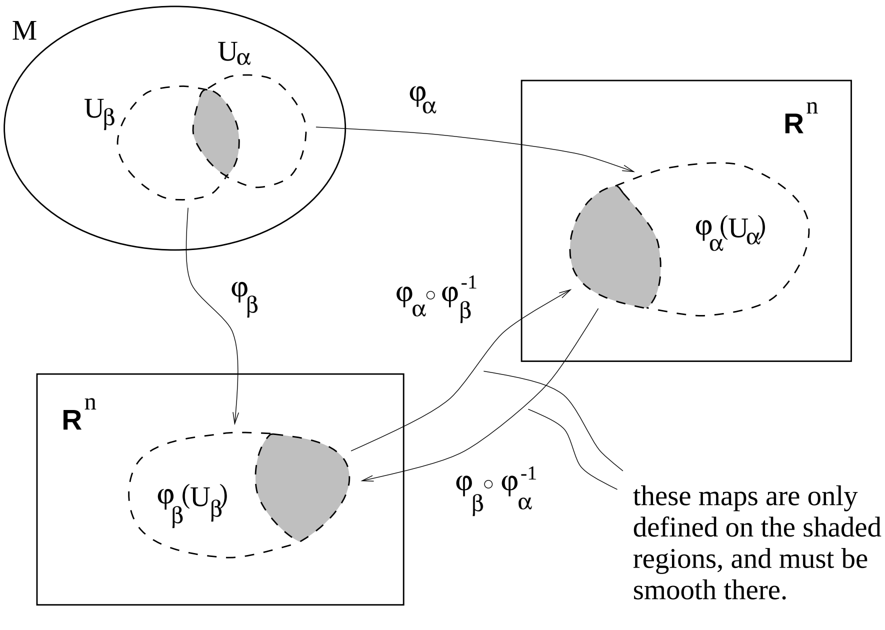
A **chart** or **coordinate system** consists of a subset U of a set M, along with a one-to-one map $\phi : U \to \mathbf{R}^n$, such that the image $\phi(U)$ is open in $\mathbf{R}$. (Any map is onto its image, so the map $\phi : U \to \phi(U)$ is invertible.) We then can say that U is an open set in M. (We have thus induced a topology on M, although we will not explore this.)



A C^∞ **atlas** is an indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$ which satisfies two conditions:

1. The union of the U_α is equal to M; that is, the U_α cover M.

2. The charts are smoothly sewn together. More precisely, if two charts overlap, $U_\alpha \cap U_\beta \neq \emptyset$, then the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$ *onto* $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$, and all of these maps must be C^∞ where they are defined. This should be clearer in pictures:

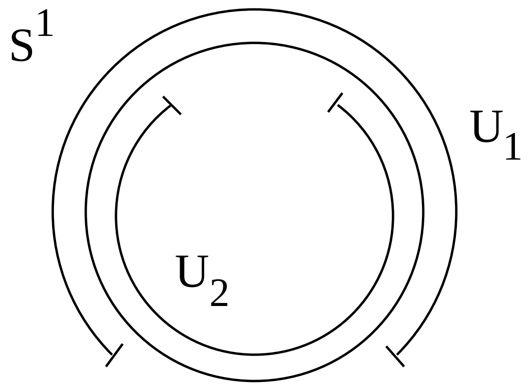


So a chart is what we normally think of as a coordinate system on some open set, and an atlas is a system of charts which are smoothly related on their overlaps.

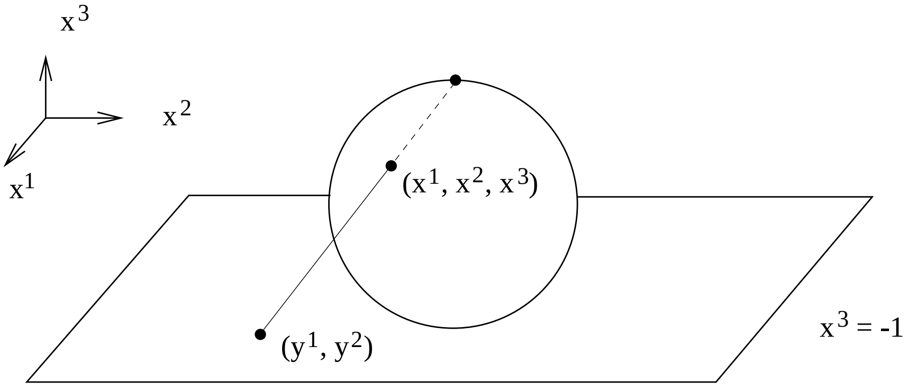
At long last, then: a C^∞ n-dimensional **manifold** (or n-manifold for short) is simply a set M along with a "maximal atlas", one that contains every possible compatible chart. (We can also replace C^∞ by C^p in all the above definitions. For our purposes the degree of differentiability of a manifold is not crucial; we will always assume that any manifold is as differentiable as necessary for the application under consideration.) The requirement that the atlas be maximal is so that two equivalent spaces equipped with different atlases don't count as different manifolds. This definition captures in formal terms our notion of a set that looks locally like $\mathbf{R}^n$. Of course we will rarely have to make use of the full power of the definition, but precision is its own reward.

One thing that is nice about our definition is that it does not rely on an embedding of the manifold in some higher-dimensional Euclidean space. In fact any n-dimensional manifold can be embedded in $\mathbf{R}^{2n}$ ("Whitney's embedding theorem"), and sometimes we will make use of this fact (such as in our definition of the sphere above). But it's important to recognize that the manifold has an individual existence independent of any embedding. We have no reason to believe, for example, that four-dimensional spacetime is stuck in some larger space. (Actually a number of people, string theorists and so forth, believe that our four-dimensional world is part of a ten- or eleven-dimensional spacetime, but as far as GR is concerned the 4-dimensional view is perfectly adequate.)

Why was it necessary to be so finicky about charts and their overlaps, rather than just covering every manifold with a single chart? Because most manifolds cannot be covered with just one chart. Consider the simplest example, S^1. There is a conventional coordinate system, $\theta : S^1 \to \mathbf{R}$, where $\theta = 0$ at the top of the circle and wraps around to 2π. However, in the definition of a chart we have required that the image $\theta(S^1)$ be open in $\mathbf{R}$. If we include either $\theta = 0$ or $\theta = 2\pi$, we have a closed interval rather than an open one; if we exclude both points, we haven't covered the whole circle. So we need at least two charts, as shown.



A somewhat more complicated example is provided by S^2, where once again no single chart will cover the manifold. A Mercator projection, traditionally used for world maps, misses both the North and South poles (as well as the International Date Line, which involves the same problem with θ that we found for S^1.) Let's take S^2 to be the set of points in $\mathbf{R}^3$ defined by $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. We can construct a chart from an open set U_1, defined to be the sphere minus the north pole, via "stereographic projection":



Thus, we draw a straight line from the north pole to the plane defined by $x^3 = -1$, and assign to the point on S^2 intercepted by the line the Cartesian coordinates (y^1, y^2) of the appropriate point on the plane. Explicitly, the map is given by

$$\phi_1(x^1, x^2, x^3) \equiv (y^1, y^2) = \left(\frac{2x^1}{1 - x^3}\ ,\ \frac{2x^2}{1 - x^3}\right)\ . \tag{2.4}$$

You are encouraged to check this for yourself. Another chart (U_2, ϕ_2) is obtained by projecting from the south pole to the plane defined by $x^3 = +1$. The resulting coordinates cover the sphere minus the south pole, and are given by

$$\phi_2(x^1, x^2, x^3) \equiv (z^1, z^2) = \left(\frac{2x^1}{1 + x^3}\ ,\ \frac{2x^2}{1 + x^3}\right)\ . \tag{2.5}$$

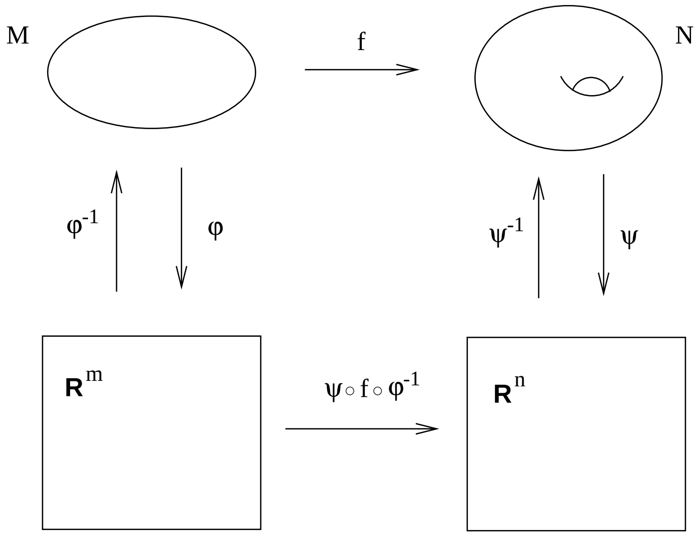
Together, these two charts cover the entire manifold, and they overlap in the region $-1 < x^3 < +1$. Another thing you can check is that the composition $\phi_2 \circ \phi_1^{-1}$ is given by

$$z^i = \frac{4y^i}{[(y^1)^2 + (y^2)^2]}\ , \tag{2.6}$$

and is C^∞ in the region of overlap. As long as we restrict our attention to this region, (2.6) is just what we normally think of as a change of coordinates.

We therefore see the necessity of charts and atlases: many manifolds cannot be covered with a single coordinate system. (Although some can, even ones with nontrivial topology. Can you think of a single good coordinate system that covers the cylinder, $S^1 \times \mathbf{R}$?) Nevertheless, it is very often most convenient to work with a single chart, and just keep track of the set of points which aren't included.

The fact that manifolds look locally like $\mathbf{R}^n$, which is manifested by the construction of coordinate charts, introduces the possibility of analysis on manifolds, including operations such as differentiation and integration. Consider two manifolds M and N of dimensions m and n, with coordinate charts ϕ on M and ψ on N. Imagine we have a function $f : M \to N$,



Just thinking of M and N as sets, we cannot nonchalantly differentiate the map f, since we don't know what such an operation means. But the coordinate charts allow us to construct the map $(\psi \circ f \circ \phi^{-1}) : \mathbf{R}^m \to \mathbf{R}^n$. (Feel free to insert the words "where the maps are defined" wherever appropriate, here and later on.) This is just a map between Euclidean spaces, and all of the concepts of advanced calculus apply. For example f, thought of as an N-valued function on M, can be differentiated to obtain $\partial f/\partial x^\mu$, where the x^μ represent $\mathbf{R}^m$. The point is that this notation is a shortcut, and what is really going on is

$$\frac{\partial f}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu}(\psi \circ f \circ \phi^{-1})(x^\mu) \ . \tag{2.7}$$

It would be far too unwieldy (not to mention pedantic) to write out the coordinate maps explicitly in every case. The shorthand notation of the left-hand-side will be sufficient for most purposes.

Having constructed this groundwork, we can now proceed to introduce various kinds of structure on manifolds. We begin with vectors and tangent spaces. In our discussion of special relativity we were intentionally vague about the definition of vectors and their relationship to the spacetime. One point that was stressed was the notion of a tangent space — the set of all vectors at a single point in spacetime. The reason for this emphasis was to remove from your minds the idea that a vector stretches from one point on the manifold to another, but instead is just an object associated with a single point. What is temporarily lost by adopting this view is a way to make sense of statements like "the vector points in the x direction" — if the tangent space is merely an abstract vector space associated with each point, it's hard to know what this should mean. Now it's time to fix the problem.

Let's imagine that we wanted to construct the tangent space at a point p in a manifold M, using only things that are intrinsic to M (no embeddings in higher-dimensional spaces etc.). One first guess might be to use our intuitive knowledge that there are objects called "tangent vectors to curves" which belong in the tangent space. We might therefore consider the set of all parameterized curves through p — that is, the space of all (nondegenerate) maps $\gamma : \mathbf{R} \to M$ such that p is in the image of γ. The temptation is to define the tangent space as simply the space of all tangent vectors to these curves at the point p. But this is obviously cheating; the tangent space T_p is supposed to be the space of vectors at p, and before we have defined this we don't have an independent notion of what "the tangent vector to a curve" is supposed to mean. In some coordinate system x^μ any curve through p defines an element of $\mathbf{R}^n$ specified by the n real numbers $dx^\mu/d\lambda$ (where λ is the parameter along the curve), but this map is clearly coordinate-dependent, which is not what we want.

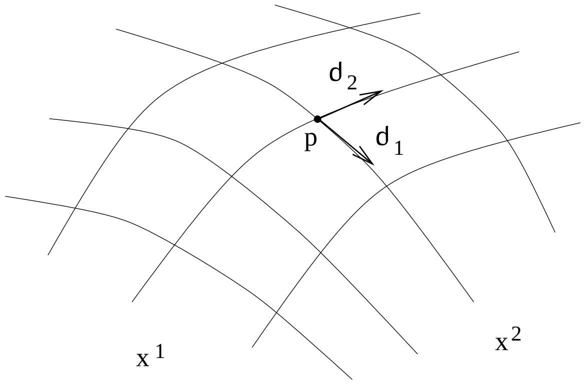
Nevertheless we are on the right track, we just have to make things independent of coordinates. To this end we define $\mathcal{F}$ to be the space of all smooth functions on M (that is, C^∞ maps $f : M \to \mathbf{R}$). Then we notice that each curve through p defines an operator on this space, the directional derivative, which maps $f \to df/d\lambda$ (at p). We will make the following claim: *the tangent space T_p can be identified with the space of directional derivative operators along curves through p.* To establish this idea we must demonstrate two things: first, that the space of directional derivatives is a vector space, and second that it is the vector space we want (it has the same dimensionality as M, yields a natural idea of a vector pointing along a certain direction, and so on).

The first claim, that directional derivatives form a vector space, seems straightforward enough. Imagine two operators $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ representing derivatives along two curves through p. There is no problem adding these and scaling by real numbers, to obtain a new operator $a\frac{d}{d\lambda} + b\frac{d}{d\eta}$. It is not immediately obvious, however, that the space closes; *i.e.*, that the resulting operator is itself a derivative operator. A good derivative operator is one that acts linearly on functions, and obeys the conventional Leibniz (product) rule on products of functions. Our new operator is manifestly linear, so we need to verify that it obeys the Leibniz rule. We have

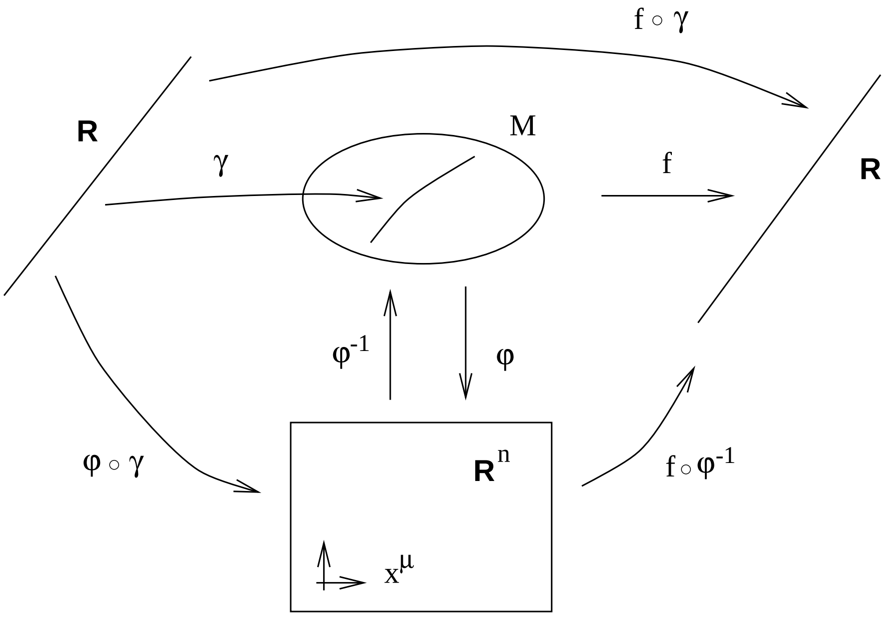
$$\begin{aligned}\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) &= af\frac{dg}{d\lambda} + ag\frac{df}{d\lambda} + bf\frac{dg}{d\eta} + bg\frac{df}{d\eta} \\ &= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f\ . \end{aligned} \tag{2.8}$$

As we had hoped, the product rule is satisfied, and the set of directional derivatives is therefore a vector space.

Is it the vector space that we would like to identify with the tangent space? The easiest way to become convinced is to find a basis for the space. Consider again a coordinate chart with coordinates x^μ. Then there is an obvious set of n directional derivatives at p, namely the partial derivatives ∂_μ at p.



We are now going to claim that the partial derivative operators $\{\partial_\mu\}$ at p form a basis for the tangent space T_p. (It follows immediately that T_p is n-dimensional, since that is the number of basis vectors.) To see this we will show that any directional derivative can be decomposed into a sum of real numbers times partial derivatives. This is in fact just the familiar expression for the components of a tangent vector, but it's nice to see it from the big-machinery approach. Consider an n-manifold M, a coordinate chart $\phi : M \to \mathbf{R}^n$, a curve $\gamma : \mathbf{R} \to M$, and a function $f : M \to \mathbf{R}$. This leads to the following tangle of maps:



If λ is the parameter along γ, we want to expand the vector/operator $\frac{d}{d\lambda}$ in terms of the partials ∂_μ. Using the chain rule (2.2), we have

$$\begin{aligned} \frac{d}{d\lambda}f &= \frac{d}{d\lambda}(f\circ\gamma) \\ &= \frac{d}{d\lambda}[(f\circ\phi^{-1})\circ(\phi\circ\gamma)] \\ &= \frac{d(\phi\circ\gamma)^\mu}{d\lambda}\frac{\partial(f\circ\phi^{-1})}{\partial x^\mu} \\ &= \frac{dx^\mu}{d\lambda}\partial_\mu f\ . \end{aligned} \tag{2.9}$$

The first line simply takes the informal expression on the left hand side and rewrites it as an honest derivative of the function $(f\circ\gamma):\mathbf{R}\rightarrow\mathbf{R}$. The second line just comes from the definition of the inverse map ϕ^{-1} (and associativity of the operation of composition). The third line is the formal chain rule (2.2), and the last line is a return to the informal notation of the start. Since the function f was arbitrary, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu\ . \tag{2.10}$$

Thus, the partials $\{\partial_\mu\}$ do indeed represent a good basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Of course, the vector represented by $\frac{d}{d\lambda}$ is one we already know; it's the tangent vector to the curve with parameter λ. Thus (2.10) can be thought of as a restatement of (1.24), where we claimed the that components of the tangent vector were simply $dx^\mu/d\lambda$. The only difference is that we are working on an arbitrary manifold, and we have specified our basis vectors to be $\hat{e}_{(\mu)} = \partial_\mu$.

This particular basis ($\hat{e}_{(\mu)} = \partial_\mu$) is known as a **coordinate basis** for T_p; it is the formalization of the notion of setting up the basis vectors to point along the coordinate axes. There is no reason why we are limited to coordinate bases when we consider tangent vectors; it is sometimes more convenient, for example, to use orthonormal bases of some sort. However, the coordinate basis is very simple and natural, and we will use it almost exclusively throughout the course.

One of the advantages of the rather abstract point of view we have taken toward vectors is that the transformation law is immediate. Since the basis vectors are $\hat{e}_{(\mu)} = \partial_\mu$, the basis vectors in some new coordinate system $x^{\mu'}$ are given by the chain rule (2.3) as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\ . \tag{2.11}$$

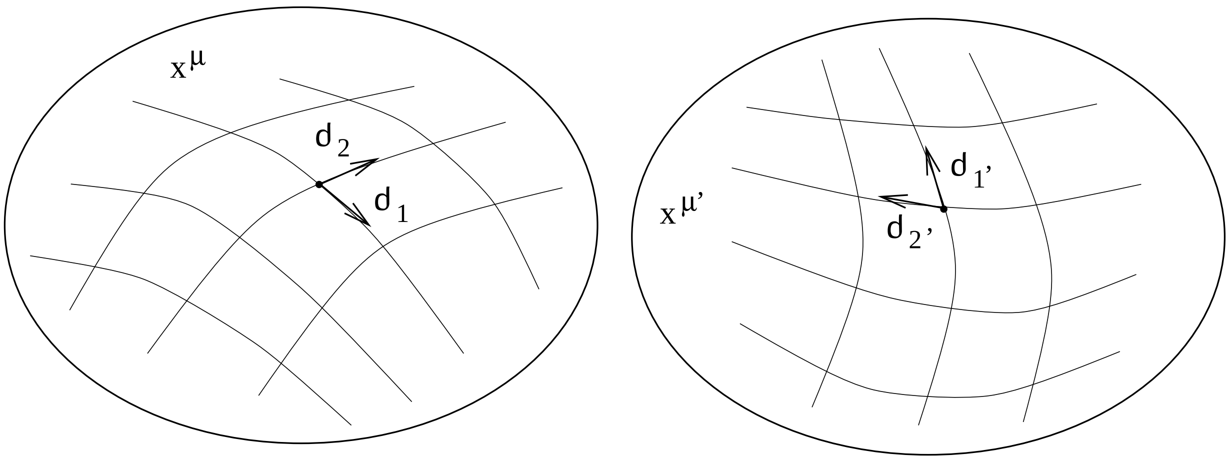
We can get the transformation law for vector components by the same technique used in flat space, demanding the the vector $V = V^\mu\partial_\mu$ be unchanged by a change of basis. We have

$$\begin{aligned} V^\mu\partial_\mu &= V^{\mu'}\partial_{\mu'} \\ &= V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\ , \end{aligned} \tag{2.12}$$

and hence (since the matrix $\partial x^{\mu'}/\partial x^{\mu}$ is the inverse of the matrix $\partial x^{\mu}/\partial x^{\mu'}$),

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \ . \tag{2.13}$$

Since the basis vectors are usually not written explicitly, the rule (2.13) for transforming components is what we call the "vector transformation law." We notice that it is compatible with the transformation of vector components in special relativity under Lorentz transformations, $V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu}$, since a Lorentz transformation is a special kind of coordinate transformation, with $x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$. But (2.13) is much more general, as it encompasses the behavior of vectors under arbitrary changes of coordinates (and therefore bases), not just linear transformations. As usual, we are trying to emphasize a somewhat subtle ontological distinction — tensor components do not change when we change coordinates, they change when we change the basis in the tangent space, but we have decided to use the coordinates to define our basis. Therefore a change of coordinates induces a change of basis:



Having explored the world of vectors, we continue to retrace the steps we took in flat space, and now consider dual vectors (one-forms). Once again the cotangent space T_p^* is the set of linear maps $\omega : T_p \rightarrow \mathbf{R}$. The canonical example of a one-form is the gradient of a function f, denoted $\mathrm{d}f$. Its action on a vector $\frac{d}{d\lambda}$ is exactly the directional derivative of the function:

$$\mathrm{d}f\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} \ . \tag{2.14}$$

It's tempting to think, "why shouldn't the function f itself be considered the one-form, and $df/d\lambda$ its action?" The point is that a one-form, like a vector, exists only at the point it is defined, and does not depend on information at other points on M. If you know a function in some neighborhood of a point you can take its derivative, but not just from knowing its value at the point; the gradient, on the other hand, encodes precisely the information necessary to take the directional derivative along any curve through p, fulfilling its role as a dual vector.

Just as the partial derivatives along coordinate axes provide a natural basis for the tangent space, the gradients of the coordinate functions x^μ provide a natural basis for the cotangent space. Recall that in flat space we constructed a basis for T_p^* by demanding that $\hat\theta^{(\mu)}(\hat e_{(\nu)}) = \delta^\mu_\nu$. Continuing the same philosophy on an arbitrary manifold, we find that (2.14) leads to

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu\ . \tag{2.15}$$

Therefore the gradients $\{\mathrm{d}x^\mu\}$ are an appropriate set of basis one-forms; an arbitrary one-form is expanded into components as $\omega = \omega_\mu\,\mathrm{d}x^\mu$.

The transformation properties of basis dual vectors and components follow from what is by now the usual procedure. We obtain, for basis one-forms,

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\,\mathrm{d}x^\mu\ , \tag{2.16}$$

and for components,

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu\ . \tag{2.17}$$

We will usually write the components ω_μ when we speak about a one-form ω.

The transformation law for general tensors follows this same pattern of replacing the Lorentz transformation matrix used in flat space with a matrix representing more general coordinate transformations. A (k,l) tensor T can be expanded

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes\mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_l}\ , \tag{2.18}$$

and under a coordinate transformation the components change according to

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\ . \tag{2.19}$$

This tensor transformation law is straightforward to remember, since there really isn't anything else it could be, given the placement of indices. However, it is often easier to transform a tensor by taking the identity of basis vectors and one-forms as partial derivatives and gradients at face value, and simply substituting in the coordinate transformation. As an example consider a symmetric $(0,2)$ tensor S on a 2-dimensional manifold, whose components in a coordinate system $(x^1 = x, x^2 = y)$ are given by

$$S_{\mu\nu} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\ . \tag{2.20}$$

This can be written equivalently as

$$\begin{aligned} S &= S_{\mu\nu}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu) \\ &= x(\mathrm{d}x)^2 + (\mathrm{d}y)^2 \ , \end{aligned} \tag{2.21}$$

where in the last line the tensor product symbols are suppressed for brevity. Now consider new coordinates

$$\begin{aligned} x' &= x^{1/3} \\ y' &= e^{x+y} \ . \end{aligned} \tag{2.22}$$

This leads directly to

$$\begin{aligned} x &= (x')^3 \\ y &= \ln(y') - (x')^3 \\ \mathrm{d}x &= 3(x')^2\,\mathrm{d}x' \\ \mathrm{d}y &= \frac{1}{y'}\,\mathrm{d}y' - 3(x')^2\,\mathrm{d}x' \ . \end{aligned} \tag{2.23}$$

We need only plug these expressions directly into (2.21) to obtain (remembering that tensor products don't commute, so $\mathrm{d}x'\,\mathrm{d}y' \neq \mathrm{d}y'\,\mathrm{d}x'$):

$$S = 9(x')^4[1+(x')^3](\mathrm{d}x')^2 - 3\frac{(x')^2}{y'}(\mathrm{d}x'\,\mathrm{d}y' + \mathrm{d}y'\,\mathrm{d}x') + \frac{1}{(y')^2}(\mathrm{d}y')^2 \ , \tag{2.24}$$

or

$$S_{\mu'\nu'} = \begin{pmatrix} 9(x')^4[1+(x')^3] & -3\frac{(x')^2}{y'} \\ -3\frac{(x')^2}{y'} & \frac{1}{(y')^2} \end{pmatrix} \ . \tag{2.25}$$

Notice that it is still symmetric. We did not use the transformation law (2.19) directly, but doing so would have yielded the same result, as you can check.

For the most part the various tensor operations we defined in flat space are unaltered in a more general setting: contraction, symmetrization, etc. There are three important exceptions: partial derivatives, the metric, and the Levi-Civita tensor. Let's look at the partial derivative first.

The unfortunate fact is that the partial derivative of a tensor is not, in general, a new tensor. The gradient, which is the partial derivative of a scalar, is an honest $(0,1)$ tensor, as we have seen. But the partial derivative of higher-rank tensors is not tensorial, as we can see by considering the partial derivative of a one-form, $\partial_\mu W_\nu$, and changing to a new coordinate system:

$$\begin{aligned} \frac{\partial}{\partial x^{\mu'}}W_{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\nu'}}W_\nu\right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\left(\frac{\partial}{\partial x^\mu}W_\nu\right) + W_\nu\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}} \ . \end{aligned} \tag{2.26}$$

The second term in the last line should not be there if $\partial_\mu W_\nu$ were to transform as a $(0,2)$ tensor. As you can see, it arises because the derivative of the transformation matrix does not vanish, as it did for Lorentz transformations in flat space.

On the other hand, the exterior derivative operator d does form an antisymmetric $(0,p+1)$ tensor when acted on a p-form. For $p=1$ we can see this from (2.26); the offending non-tensorial term can be written

$$W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} = W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} \ . \tag{2.27}$$

This expression is symmetric in μ' and ν', since partial derivatives commute. But the exterior derivative is defined to be the antisymmetrized partial derivative, so this term vanishes (the antisymmetric part of a symmetric expression is zero). We are then left with the correct tensor transformation law; extension to arbitrary p is straightforward. So the exterior derivative is a legitimate tensor operator; it is not, however, an adequate substitute for the partial derivative, since it is only defined on forms. In the next section we will define a covariant derivative, which can be thought of as the extension of the partial derivative to arbitrary manifolds.

The metric tensor is such an important object in curved space that it is given a new symbol, $g_{\mu\nu}$ (while $\eta_{\mu\nu}$ is reserved specifically for the Minkowski metric). There are few restrictions on the components of $g_{\mu\nu}$, other than that it be a symmetric $(0,2)$ tensor. It is usually taken to be non-degenerate, meaning that the determinant $g = |g_{\mu\nu}|$ doesn't vanish. This allows us to define the inverse metric $g^{\mu\nu}$ via

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \ . \tag{2.28}$$

The symmetry of $g_{\mu\nu}$ implies that $g^{\mu\nu}$ is also symmetric. Just as in special relativity, the metric and its inverse may be used to raise and lower indices on tensors.

It will take several weeks to fully appreciate the role of the metric in all of its glory, but for purposes of inspiration we can list the various uses to which $g_{\mu\nu}$ will be put: (1) the metric supplies a notion of "past" and "future"; (2) the metric allows the computation of path length and proper time; (3) the metric determines the "shortest distance" between two points (and therefore the motion of test particles); (4) the metric replaces the Newtonian gravitational field ϕ; (5) the metric provides a notion of locally inertial frames and therefore a sense of "no rotation"; (6) the metric determines causality, by defining the speed of light faster than which no signal can travel; (7) the metric replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics; and so on. Obviously these ideas are not all completely independent, but we get some sense of the importance of this tensor.

In our discussion of path lengths in special relativity we (somewhat handwavingly) introduced the line element as $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, which was used to get the length of a path.

Of course now that we know that $\mathrm{d}x^\mu$ is really a basis dual vector, it becomes natural to use the terms "metric" and "line element" interchangeably, and write

$$ds^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \ . \tag{2.29}$$

(To be perfectly consistent we should write this as "g", and sometimes will, but more often than not g is used for the determinant $|g_{\mu\nu}|$.) For example, we know that the Euclidean line element in a three-dimensional space with Cartesian coordinates is

$$ds^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \ . \tag{2.30}$$

We can now change to any coordinate system we choose. For example, in spherical coordinates we have

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \ , \end{aligned} \tag{2.31}$$

which leads directly to

$$ds^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \ . \tag{2.32}$$

Obviously the components of the metric look different than those in Cartesian coordinates, but all of the properties of the space remain unaltered.

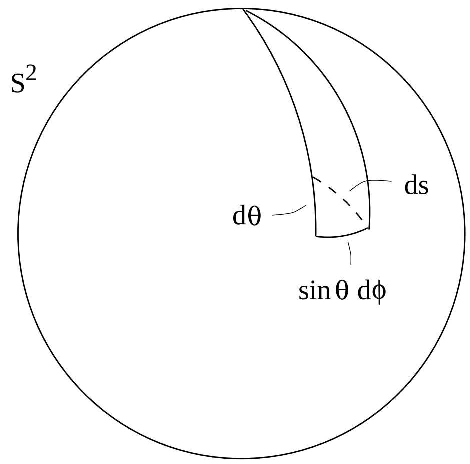
Perhaps this is a good time to note that most references are not sufficiently picky to distinguish between "dx", the informal notion of an infinitesimal displacement, and "$\mathrm{d}x$", the rigorous notion of a basis one-form given by the gradient of a coordinate function. In fact our notation "ds^2" does not refer to the exterior derivative of anything, or the square of anything; it's just conventional shorthand for the metric tensor. On the other hand, "$(\mathrm{d}x)^2$" refers specifically to the $(0,2)$ tensor $\mathrm{d}x \otimes \mathrm{d}x$.

A good example of a space with curvature is the two-sphere, which can be thought of as the locus of points in $\mathbf{R}^3$ at distance 1 from the origin. The metric in the (θ, ϕ) coordinate system comes from setting $r = 1$ and $\mathrm{d}r = 0$ in (2.32):

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \ . \tag{2.33}$$

This is completely consistent with the interpretation of ds as an infinitesimal length, as illustrated in the figure.

As we shall see, the metric tensor contains all the information we need to describe the curvature of the manifold (at least in Riemannian geometry; we will actually indicate somewhat more general approaches). In Minkowski space we can choose coordinates in which the components of the metric are constant; but it should be clear that the existence of curvature



is more subtle than having the metric depend on the coordinates, since in the example above we showed how the metric in flat Euclidean space in spherical coordinates is a function of r and θ. Later, we shall see that constancy of the metric components is sufficient for a space to be flat, and in fact there always exists a coordinate system on any flat space in which the metric is constant. But we might not want to work in such a coordinate system, and we might not even know how to find it; therefore we will want a more precise characterization of the curvature, which will be introduced down the road.

A useful characterization of the metric is obtained by putting $g_{\mu\nu}$ into its **canonical form**. In this form the metric components become

$$g_{\mu\nu} = \text{ diag } (-1, -1, \ldots, -1, +1, +1, \ldots, +1, 0, 0, \ldots, 0) \ , \tag{2.34}$$

where "diag" means a diagonal matrix with the given elements. If n is the dimension of the manifold, s is the number of $+1$'s in the canonical form, and t is the number of -1's, then $s - t$ is the **signature** of the metric (the difference in the number of minus and plus signs), and $s + t$ is the **rank** of the metric (the number of nonzero eigenvalues). If a metric is continuous, the rank and signature of the metric tensor field are the same at every point, and if the metric is nondegenerate the rank is equal to the dimension n. We will always deal with continuous, nondegenerate metrics. If all of the signs are positive ($t = 0$) the metric is called **Euclidean** or **Riemannian** (or just "positive definite"), while if there is a single minus ($t = 1$) it is called **Lorentzian** or **pseudo-Riemannian**, and any metric with some $+1$'s and some -1's is called "indefinite." (So the word "Euclidean" sometimes means that the space is flat, and sometimes doesn't, but always means that the canonical form is strictly positive; the terminology is unfortunate but standard.) The spacetimes of interest in general relativity have Lorentzian metrics.

We haven't yet demonstrated that it is always possible to but the metric into canonical form. In fact it is always possible to do so at some point $p \in M$, but in general it will only be possible at that single point, not in any neighborhood of p. Actually we can do slightly better than this; it turns out that at any point p there exists a coordinate system in which $g_{\mu\nu}$ takes its canonical form and the first derivatives $\partial_\sigma g_{\mu\nu}$ all vanish (while the second derivatives $\partial_\rho\partial_\sigma g_{\mu\nu}$ cannot be made to all vanish). Such coordinates are known as **Riemann normal coordinates**, and the associated basis vectors constitute a **local Lorentz frame**. Notice that in Riemann normal coordinates (or RNC's) the metric at p looks like that of flat space "to first order." This is the rigorous notion of the idea that "small enough regions of spacetime look like flat (Minkowski) space." (Also, there is no difficulty in simultaneously constructing sets of basis vectors at every point in M such that the metric takes its canonical form; the problem is that in general this will not be a *coordinate* basis, and there will be no way to make it into one.)

We won't consider the detailed proof of this statement; it can be found in Schutz, pp. 158-160, where it goes by the name of the "local flatness theorem." (He also calls local Lorentz frames "momentarily comoving reference frames," or MCRF's.) It is useful to see a sketch of the proof, however, for the specific case of a Lorentzian metric in four dimensions. The idea is to consider the transformation law for the metric

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \ , \tag{2.35}$$

and expand both sides in Taylor series in the sought-after coordinates $x^{\mu'}$. The expansion of the old coordinates x^μ looks like

$$x^\mu = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)_p x^{\mu'} + \frac{1}{2}\left(\frac{\partial^2 x^\mu}{\partial x^{\mu'_1}\partial x^{\mu'_2}}\right)_p x^{\mu'_1}x^{\mu'_2} + \frac{1}{6}\left(\frac{\partial^3 x^\mu}{\partial x^{\mu'_1}\partial x^{\mu'_2}\partial x^{\mu'_3}}\right)_p x^{\mu'_1}x^{\mu'_2}x^{\mu'_3} + \cdots \ , \tag{2.36}$$

with the other expansions proceeding along the same lines. (For simplicity we have set $x^\mu(p) = x^{\mu'}(p) = 0$.) Then, using some extremely schematic notation, the expansion of (2.35) to second order is

$$\begin{aligned} &(g')_p + (\partial' g')_p\, x' + (\partial'\partial' g')_p\, x'x' \\ &= \left(\frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'} g\right)_p + \left(\frac{\partial x}{\partial x'}\frac{\partial^2 x}{\partial x'\partial x'} g + \frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'}\partial' g\right)_p x' \\ &\quad + \left(\frac{\partial x}{\partial x'}\frac{\partial^3 x}{\partial x'\partial x'\partial x'} g + \frac{\partial^2 x}{\partial x'\partial x'}\frac{\partial^2 x}{\partial x'\partial x'} g + \frac{\partial x}{\partial x'}\frac{\partial^2 x}{\partial x'\partial x'}\partial' g + \frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'}\partial'\partial' g\right)_p x'x' \end{aligned} \tag{2.37}$$

We can set terms of equal order in x' on each side equal to each other. Therefore, the components $g_{\mu'\nu'}(p)$, 10 numbers in all (to describe a symmetric two-index tensor), are determined by the matrix $(\partial x^\mu/\partial x^{\mu'})_p$. This is a 4×4 matrix with no constraints; thus, 16 numbers we are free to choose. Clearly this is enough freedom to put the 10 numbers of $g_{\mu'\nu'}(p)$ into canonical form, at least as far as having enough degrees of freedom is concerned.

(In fact there are some limitations — if you go through the procedure carefully, you find for example that you cannot change the signature and rank.) The six remaining degrees of freedom can be interpreted as exactly the six parameters of the Lorentz group; we know that these leave the canonical form unchanged. At first order we have the derivatives $\partial_{\sigma'} g_{\mu'\nu'}(p)$, four derivatives of ten components for a total of 40 numbers. But looking at the right hand side of (2.37) we see that we now have the additional freedom to choose $(\partial^2 x^\mu/\partial x^{\mu'_1}\partial x^{\mu'_2})_p$. In this set of numbers there are 10 independent choices of the indices μ'_1 and μ'_2 (it's symmetric, since partial derivatives commute) and four choices of μ, for a total of 40 degrees of freedom. This is precisely the amount of choice we need to determine all of the first derivatives of the metric, which we can therefore set to zero. At second order, however, we are concerned with $\partial_{\rho'}\partial_{\sigma'} g_{\mu'\nu'}(p)$; this is symmetric in ρ' and σ' as well as μ' and ν', for a total of $10\times 10 = 100$ numbers. Our ability to make additional choices is contained in $(\partial^3 x^\mu/\partial x^{\mu'_1}\partial x^{\mu'_2}\partial x^{\mu'_3})_p$. This is symmetric in the three lower indices, which gives 20 possibilities, times four for the upper index gives us 80 degrees of freedom — 20 fewer than we require to set the second derivatives of the metric to zero. So in fact we cannot make the second derivatives vanish; the deviation from flatness must therefore be measured by the 20 coordinate-independent degrees of freedom representing the second derivatives of the metric tensor field. We will see later how this comes about, when we characterize curvature using the Riemann tensor, which will turn out to have 20 independent components.

The final change we have to make to our tensor knowledge now that we have dropped the assumption of flat space has to do with the Levi-Civita tensor, $\epsilon_{\mu_1\mu_2\cdots\mu_n}$. Remember that the flat-space version of this object, which we will now denote by $\tilde\epsilon_{\mu_1\mu_2\cdots\mu_n}$, was defined as

$$\tilde\epsilon_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 \text{ if } \mu_1\mu_2\cdots\mu_n \text{ is an even permutation of } 01\cdots(n-1)\ , \\ -1 \text{ if } \mu_1\mu_2\cdots\mu_n \text{ is an odd permutation of } 01\cdots(n-1)\ , \\ 0 \text{ otherwise}\ . \end{cases} \tag{2.38}$$

We will now define the **Levi-Civita symbol** to be exactly this $\tilde\epsilon_{\mu_1\mu_2\cdots\mu_n}$ — that is, an object with n indices which has the components specified above *in any coordinate system.* This is called a "symbol," of course, because it is not a tensor; it is defined not to change under coordinate transformations. We can relate its behavior to that of an ordinary tensor by first noting that, given some $n\times n$ matrix $M^\mu{}_{\mu'}$, the determinant $|M|$ obeys

$$\tilde\epsilon_{\mu'_1\mu'_2\cdots\mu'_n}|M| = \tilde\epsilon_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}{}_{\mu'_1} M^{\mu_2}{}_{\mu'_2}\cdots M^{\mu_n}{}_{\mu'_n}\ . \tag{2.39}$$

This is just a true fact about the determinant which you can find in a sufficiently enlightened linear algebra book. If follows that, setting $M^\mu{}_{\mu'} = \partial x^\mu/\partial x^{\mu'}$, we have

$$\tilde\epsilon_{\mu'_1\mu'_2\cdots\mu'_n} = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| \tilde\epsilon_{\mu_1\mu_2\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\frac{\partial x^{\mu_2}}{\partial x^{\mu'_2}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}\ . \tag{2.40}$$

This is close to the tensor transformation law, except for the determinant out front. Objects which transform in this way are known as **tensor densities**. Another example is given by the determinant of the metric, $g = |g_{\mu\nu}|$. It's easy to check (by taking the determinant of both sides of (2.35)) that under a coordinate transformation we get

$$g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-2} g(x^{\mu}) \ . \tag{2.41}$$

Therefore g is also not a tensor; it transforms in a way similar to the Levi-Civita symbol, except that the Jacobian is raised to the -2 power. The power to which the Jacobian is raised is known as the **weight** of the tensor density; the Levi-Civita symbol is a density of weight 1, while g is a (scalar) density of weight -2.

However, we don't like tensor densities, we like tensors. There is a simple way to convert a density into an honest tensor — multiply by $|g|^{w/2}$, where w is the weight of the density (the absolute value signs are there because $g < 0$ for Lorentz metrics). The result will transform according to the tensor transformation law. Therefore, for example, we can define the Levi-Civita tensor as

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} \ . \tag{2.42}$$

It is this tensor which is used in the definition of the Hodge dual, (1.87), which is otherwise unchanged when generalized to arbitrary manifolds. Since this is a real tensor, we can raise indices, etc. Sometimes people define a version of the Levi-Civita symbol with upper indices, $\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$, whose components are numerically equal to the symbol with lower indices. This turns out to be a density of weight -1, and is related to the tensor with upper indices by

$$\epsilon^{\mu_1\mu_2\cdots\mu_n} = \ \mathrm{sgn}(g)\frac{1}{\sqrt{|g|}}\,\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n} \ . \tag{2.43}$$

As an aside, we should come clean and admit that, even with the factor of $\sqrt{|g|}$, the Levi-Civita tensor is in some sense not a true tensor, because on some manifolds it cannot be globally defined. Those on which it can be defined are called **orientable**, and we will deal exclusively with orientable manifolds in this course. An example of a non-orientable manifold is the Möbius strip; see Schutz's *Geometrical Methods in Mathematical Physics* (or a similar text) for a discussion.

One final appearance of tensor densities is in integration on manifolds. We will not do this subject justice, but at least a casual glance is necessary. You have probably been exposed to the fact that in ordinary calculus on $\mathbf{R}^n$ the volume element $d^n x$ picks up a factor of the Jacobian under change of coordinates:

$$d^n x' = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| d^n x \ . \tag{2.44}$$

There is actually a beautiful explanation of this formula from the point of view of differential forms, which arises from the following fact: *on an n-dimensional manifold, the integrand is properly understood as an n-form.* The naive volume element $d^n x$ is itself a density rather than an n-form, but there is no difficulty in using it to construct a real n-form. To see how this works, we should make the identification

$$d^n x \leftrightarrow \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \ . \tag{2.45}$$

The expression on the right hand side can be misleading, because it looks like a tensor (an n-form, actually) but is really a density. Certainly if we have two functions f and g on M, then $\mathrm{d}f$ and $\mathrm{d}g$ are one-forms, and $\mathrm{d}f \wedge \mathrm{d}g$ is a two-form. But we would like to interpret the right hand side of (2.45) as a coordinate-dependent object which, in the x^μ coordinate system, acts like $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$. This sounds tricky, but in fact it's just an ambiguity of notation, and in practice we will just use the shorthand notation "$d^n x$".

To justify this song and dance, let's see how (2.45) changes under coordinate transformations. First notice that the definition of the wedge product allows us to write

$$\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1\cdots\mu_n}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \ , \tag{2.46}$$

since both the wedge product and the Levi-Civita symbol are completely antisymmetric. Under a coordinate transformation $\tilde{\epsilon}_{\mu_1\cdots\mu_n}$ stays the same while the one-forms change according to (2.16), leading to

$$\begin{aligned} \tilde{\epsilon}_{\mu_1\cdots\mu_n}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} &= \tilde{\epsilon}_{\mu_1\cdots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}\,\mathrm{d}x^{\mu'_1} \wedge \cdots \wedge \mathrm{d}x^{\mu'_n} \\ &= \left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right|\tilde{\epsilon}_{\mu'_1\cdots\mu'_n}\,\mathrm{d}x^{\mu'_1} \wedge \cdots \wedge \mathrm{d}x^{\mu'_n} \ . \end{aligned} \tag{2.47}$$

Multiplying by the Jacobian on both sides recovers (2.44).

It is clear that the naive volume element $d^n x$ transforms as a density, not a tensor, but it is straightforward to construct an invariant volume element by multiplying by $\sqrt{|g|}$:

$$\sqrt{|g'|}\,\mathrm{d}x^{0'} \wedge \cdots \wedge \mathrm{d}x^{(n-1)'} = \sqrt{|g|}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \ , \tag{2.48}$$

which is of course just $(n!)^{-1}\epsilon_{\mu_1\cdots\mu_n}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$. In the interest of simplicity we will usually write the volume element as $\sqrt{|g|}\,d^n x$, rather than as the explicit wedge product $\sqrt{|g|}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$; it will be enough to keep in mind that it's supposed to be an n-form.

As a final aside to finish this section, let's consider one of the most elegant and powerful theorems of differential geometry: Stokes's theorem. This theorem is the generalization of the fundamental theorem of calculus, $\int_b^a dx = a - b$. Imagine that we have an n-manifold

M with boundary ∂M, and an $(n-1)$-form ω on M. (We haven't discussed manifolds with boundaries, but the idea is obvious; M could for instance be the interior of an $(n-1)$-dimensional closed surface ∂M.) Then $\mathrm{d}\omega$ is an n-form, which can be integrated over M, while ω itself can be integrated over ∂M. Stokes's theorem is then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \ . \tag{2.49}$$

You can convince yourself that different special cases of this theorem include not only the fundamental theorem of calculus, but also the theorems of Green, Gauss, and Stokes, familiar from vector calculus in three dimensions.

3 Curvature

In our discussion of manifolds, it became clear that there were various notions we could talk about as soon as the manifold was defined; we could define functions, take their derivatives, consider parameterized paths, set up tensors, and so on. Other concepts, such as the volume of a region or the length of a path, required some additional piece of structure, namely the introduction of a metric. It would be natural to think of the notion of "curvature", which we have already used informally, is something that depends on the metric. Actually this turns out to be not quite true, or at least incomplete. In fact there is one additional structure we need to introduce — a "connection" — which is characterized by the curvature. We will show how the existence of a metric implies a certain connection, whose curvature may be thought of as that of the metric.

The connection becomes necessary when we attempt to address the problem of the partial derivative not being a good tensor operator. What we would like is a covariant derivative; that is, an operator which reduces to the partial derivative in flat space with Cartesian coordinates, but transforms as a tensor on an arbitrary manifold. It is conventional to spend a certain amount of time motivating the introduction of a covariant derivative, but in fact the need is obvious; equations such as $\partial_\mu T^{\mu\nu} = 0$ are going to have to be generalized to curved space somehow. So let's agree that a covariant derivative would be a good thing to have, and go about setting it up.

In flat space in Cartesian coordinates, the partial derivative operator ∂_μ is a map from (k, l) tensor fields to $(k, l+1)$ tensor fields, which acts linearly on its arguments and obeys the Leibniz rule on tensor products. All of this continues to be true in the more general situation we would now like to consider, but the map provided by the partial derivative depends on the coordinate system used. We would therefore like to define a **covariant derivative** operator ∇ to perform the functions of the partial derivative, but in a way independent of coordinates. We therefore require that ∇ be a map from (k, l) tensor fields to $(k, l+1)$ tensor fields which has these two properties:

1. linearity: $\nabla(T + S) = \nabla T + \nabla S$;

2. Leibniz (product) rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

If ∇ is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation. That is, to take the covariant derivative we first take the partial derivative, and then apply a correction to make the result covariant. (We aren't going to prove this reasonable-sounding statement, but Wald goes into detail if you are interested.)

Let's consider what this means for the covariant derivative of a vector V^ν. It means that, for each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a matrix $(\Gamma_\mu)^\rho{}_\sigma$ (an $n \times n$ matrix, where n is the dimensionality of the manifold, for each μ). In fact the parentheses are usually dropped and we write these matrices, known as the **connection coefficients**, with haphazard index placement as $\Gamma^\rho_{\mu\sigma}$. We therefore have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \ . \tag{3.1}$$

Notice that in the second term the index originally on V has moved to the Γ, and a new index is summed over. If this is the expression for the covariant derivative of a vector in terms of the partial derivative, we should be able to determine the transformation properties of $\Gamma^\nu_{\mu\lambda}$ by demanding that the left hand side be a $(1,1)$ tensor. That is, we want the transformation law to be

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \ . \tag{3.2}$$

Let's look at the left side first; we can expand it using (3.1) and then transform the parts that we understand:

$$\begin{aligned} \nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \ . \end{aligned} \tag{3.3}$$

The right side, meanwhile, can likewise be expanded:

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda \ . \tag{3.4}$$

These last two expressions are to be equated; the first terms in each are identical and therefore cancel, so we have

$$\Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\lambda \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\lambda} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda \ , \tag{3.5}$$

where we have changed a dummy index from ν to λ. This equation must be true for any vector V^λ, so we can eliminate that on both sides. Then the connection coefficients in the primed coordinates may be isolated by multiplying by $\partial x^\lambda / \partial x^{\lambda'}$. The result is

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \ . \tag{3.6}$$

This is not, of course, the tensor transformation law; the second term on the right spoils it. That's okay, because *the connection coefficients are not the components of a tensor.* They are purposefully constructed to be non-tensorial, but in such a way that the combination (3.1) transforms as a tensor — the extra terms in the transformation of the partials and the Γ's exactly cancel. This is why we are not so careful about index placement on the connection coefficients; they are not a tensor, and therefore you should try not to raise and lower their indices.

What about the covariant derivatives of other sorts of tensors? By similar reasoning to that used for vectors, the covariant derivative of a one-form can also be expressed as a partial derivative plus some linear transformation. But there is no reason as yet that the matrices representing this transformation should be related to the coefficients $\Gamma^{\nu}_{\mu\lambda}$. In general we could write something like

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \widetilde{\Gamma}^{\lambda}_{\mu\nu}\omega_\lambda \ , \tag{3.7}$$

where $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$ is a new set of matrices for each μ. (Pay attention to where all of the various indices go.) It is straightforward to derive that the transformation properties of $\widetilde{\Gamma}$ must be the same as those of Γ, but otherwise no relationship has been established. To do so, we need to introduce two new properties that we would like our covariant derivative to have (in addition to the two above):

3. commutes with contractions: $\nabla_\mu (T^{\lambda}{}_{\lambda\rho}) = (\nabla T)_\mu{}^{\lambda}{}_{\lambda\rho}$,

4. reduces to the partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$.

There is no way to "derive" these properties; we are simply demanding that they be true as part of the definition of a covariant derivative.

Let's see what these new properties imply. Given some one-form field ω_μ and vector field V^μ, we can take the covariant derivative of the scalar defined by $\omega_\lambda V^\lambda$ to get

$$\begin{aligned} \nabla_\mu(\omega_\lambda V^\lambda) &= (\nabla_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \widetilde{\Gamma}^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \omega_\lambda \Gamma^{\lambda}_{\mu\rho} V^\rho \ . \end{aligned} \tag{3.8}$$

But since $\omega_\lambda V^\lambda$ is a scalar, this must also be given by the partial derivative:

$$\begin{aligned} \nabla_\mu(\omega_\lambda V^\lambda) &= \partial_\mu(\omega_\lambda V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) \ . \end{aligned} \tag{3.9}$$

This can only be true if the terms in (3.8) with connection coefficients cancel each other; that is, rearranging dummy indices, we must have

$$0 = \widetilde{\Gamma}^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda + \Gamma^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda \ . \tag{3.10}$$

But both ω_σ and V^λ are completely arbitrary, so

$$\widetilde{\Gamma}^{\sigma}_{\mu\lambda} = -\Gamma^{\sigma}_{\mu\lambda} \ . \tag{3.11}$$

The two extra conditions we have imposed therefore allow us to express the covariant derivative of a one-form using the same connection coefficients as were used for the vector, but now with a minus sign (and indices matched up somewhat differently):

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \ . \tag{3.12}$$

It should come as no surprise that the connection coefficients encode all of the information necessary to take the covariant derivative of a tensor of arbitrary rank. The formula is quite straightforward; for each upper index you introduce a term with a single $+\Gamma$, and for each lower index a term with a single $-\Gamma$:

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &\quad +\Gamma^{\mu_1}_{\sigma\lambda}\,T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda}\,T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ &\quad -\Gamma^{\lambda}_{\sigma\nu_1}T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots \ .\end{aligned} \tag{3.13}$$

This is the general expression for the covariant derivative. You can check it yourself; it comes from the set of axioms we have established, and the usual requirements that tensors of various sorts be coordinate-independent entities. Sometimes an alternative notation is used; just as commas are used for partial derivatives, semicolons are used for covariant ones:

$$\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \equiv T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l;\sigma} \ . \tag{3.14}$$

Once again, I'm not a big fan of this notation.

To define a covariant derivative, then, we need to put a "connection" on our manifold, which is specified in some coordinate system by a set of coefficients $\Gamma^\lambda_{\mu\nu}$ ($n^3 = 64$ independent components in $n = 4$ dimensions) which transform according to (3.6). (The name "connection" comes from the fact that it is used to transport vectors from one tangent space to another, as we will soon see.) There are evidently a large number of connections we could define on any manifold, and each of them implies a distinct notion of covariant differentiation. In general relativity this freedom is not a big concern, because it turns out that every metric defines a unique connection, which is the one used in GR. Let's see how that works.

The first thing to notice is that the difference of two connections is a $(1,2)$ tensor. If we have two sets of connection coefficients, $\Gamma^\lambda_{\mu\nu}$ and $\widehat{\Gamma}^\lambda_{\mu\nu}$, their difference $S_{\mu\nu}{}^\lambda = \Gamma^\lambda_{\mu\nu} - \widehat{\Gamma}^\lambda_{\mu\nu}$ (notice index placement) transforms as

$$\begin{aligned}S_{\mu'\nu'}{}^{\lambda'} &= \Gamma^{\lambda'}_{\mu'\nu'} - \widehat{\Gamma}^{\lambda'}_{\mu'\nu'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\widehat{\Gamma}^\lambda_{\mu\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}(\Gamma^\lambda_{\mu\nu} - \widehat{\Gamma}^\lambda_{\mu\nu}) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}S_{\mu\nu}{}^\lambda \ .\end{aligned} \tag{3.15}$$

This is just the tensor transormation law, so $S_{\mu\nu}{}^{\lambda}$ is indeed a tensor. This implies that any set of connections can be expressed as some fiducial connection plus a tensorial correction.

Next notice that, given a connection specified by $\Gamma^{\lambda}_{\mu\nu}$, we can immediately form another connection simply by permuting the lower indices. That is, the set of coefficients $\Gamma^{\lambda}_{\nu\mu}$ will also transform according to (3.6) (since the partial derivatives appearing in the last term can be commuted), so they determine a distinct connection. There is thus a tensor we can associate with any given connection, known as the **torsion tensor**, defined by

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]} \ . \tag{3.16}$$

It is clear that the torsion is antisymmetric its lower indices, and a connection which is symmetric in its lower indices is known as "torsion-free."

We can now define a unique connection on a manifold with a metric $g_{\mu\nu}$ by introducing two additional properties:

- torsion-free: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$.
- metric compatibility: $\nabla_{\rho} g_{\mu\nu} = 0$.

A connection is metric compatible if the covariant derivative of the metric with respect to that connection is everywhere zero. This implies a couple of nice properties. First, it's easy to show that the inverse metric also has zero covariant derivative,

$$\nabla_{\rho} g^{\mu\nu} = 0 \ . \tag{3.17}$$

Second, a metric-compatible covariant derivative commutes with raising and lowering of indices. Thus, for some vector field V^{λ},

$$g_{\mu\lambda}\nabla_{\rho}V^{\lambda} = \nabla_{\rho}(g_{\mu\lambda}V^{\lambda}) = \nabla_{\rho}V_{\mu} \ . \tag{3.18}$$

With non-metric-compatible connections one must be very careful about index placement when taking a covariant derivative.

Our claim is therefore that there is exactly one torsion-free connection on a given manifold which is compatible with some given metric on that manifold. We do not want to make these two requirements part of the definition of a covariant derivative; they simply single out one of the many possible ones.

We can demonstrate both existence and uniqueness by deriving a manifestly unique expression for the connection coefficients in terms of the metric. To accomplish this, we expand out the equation of metric compatibility for three different permutations of the indices:

$$\nabla_{\rho} g_{\mu\nu} \ = \ \partial_{\rho} g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0$$

$$
\begin{aligned}
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \\
\nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \ .
\end{aligned}
\tag{3.19}
$$

We subtract the second and third of these from the first, and use the symmetry of the connection to obtain

$$
\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0 \ . \tag{3.20}
$$

It is straightforward to solve this for the connection by multiplying by $g^{\sigma\rho}$. The result is

$$
\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \ . \tag{3.21}
$$

This is one of the most important formulas in this subject; commit it to memory. Of course, we have only proved that if a metric-compatible and torsion-free connection exists, it must be of the form (3.21); you can check for yourself (for those of you without enough tedious computation in your lives) that the right hand side of (3.21) transforms like a connection.

This connection we have derived from the metric is the one on which conventional general relativity is based (although we will keep an open mind for a while longer). It is known by different names: sometimes the **Christoffel** connection, sometimes the **Levi-Civita** connection, sometimes the **Riemannian** connection. The associated connection coefficients are sometimes called **Christoffel symbols** and written as $\left\{ {\sigma \atop \mu\nu} \right\}$; we will sometimes call them Christoffel symbols, but we won't use the funny notation. The study of manifolds with metrics and their associated connections is called "Riemannian geometry." As far as I can tell the study of more general connections can be traced back to Cartan, but I've never heard it called "Cartanian geometry."

Before putting our covariant derivatives to work, we should mention some miscellaneous properties. First, let's emphasize again that the connection does not *have* to be constructed from the metric. In ordinary flat space there is an implicit connection we use all the time — the Christoffel connection constructed from the flat metric. But we could, if we chose, use a different connection, while keeping the metric flat. Also notice that the coefficients of the Christoffel connection in flat space will vanish in Cartesian coordinates, but not in curvilinear coordinate systems. Consider for example the plane in polar coordinates, with metric

$$
ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \ . \tag{3.22}
$$

The nonzero components of the inverse metric are readily found to be $g^{rr} = 1$ and $g^{\theta\theta} = r^{-2}$. (Notice that we use r and θ as indices in an obvious notation.) We can compute a typical connection coefficient:

$$
\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2} g^{r\rho}(\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr}) \\
&= \frac{1}{2} g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr})
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}g^{r\theta}(\partial_r g_{r\theta} + \partial_r g_{\theta r} - \partial_\theta g_{rr}) \\
= & \ \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0)(0+0-0) \\
= & \ 0\ .
\end{aligned} \tag{3.23}
$$

Sadly, it vanishes. But not all of them do:

$$
\begin{aligned}
\Gamma^r_{\theta\theta} &= \frac{1}{2}g^{r\rho}(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}) \\
&= \frac{1}{2}g^{rr}(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}) \\
&= \frac{1}{2}(1)(0+0-2r) \\
&= -r\ .
\end{aligned} \tag{3.24}
$$

Continuing to turn the crank, we eventually find

$$
\begin{aligned}
\Gamma^r_{\theta r} &= \Gamma^r_{r\theta} = 0 \\
\Gamma^\theta_{rr} &= 0 \\
\Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r} \\
\Gamma^\theta_{\theta\theta} &= 0\ .
\end{aligned} \tag{3.25}
$$

The existence of nonvanishing connection coefficients in curvilinear coordinate systems is the ultimate cause of the formulas for the divergence and so on that you find in books on electricity and magnetism.

Contrariwise, even in a curved space it is still possible to make the Christoffel symbols vanish at any one point. This is just because, as we saw in the last section, we can always make the first derivative of the metric vanish at a point; so by (3.21) the connection coefficients derived from this metric will also vanish. Of course this can only be established at a point, not in some neighborhood of the point.

Another useful property is that the formula for the divergence of a vector (with respect to the Christoffel connection) has a simplified form. The covariant divergence of V^μ is given by

$$
\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda}V^\lambda\ . \tag{3.26}
$$

It's easy to show (see pp. 106-108 of Weinberg) that the Christoffel connection satisfies

$$
\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda\sqrt{|g|}\ , \tag{3.27}
$$

and we therefore obtain

$$
\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu)\ . \tag{3.28}
$$

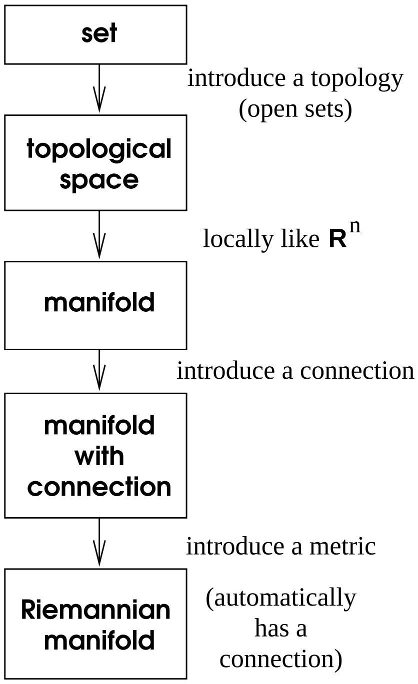
There are also formulas for the divergences of higher-rank tensors, but they are generally not such a great simplification.

As the last factoid we should mention about connections, let us emphasize (once more) that the exterior derivative is a well-defined tensor in the absence of any connection. The reason this needs to be emphasized is that, if you happen to be using a symmetric (torsion-free) connection, the exterior derivative (defined to be the antisymmetrized partial derivative) happens to be equal to the antisymmetrized covariant derivative:

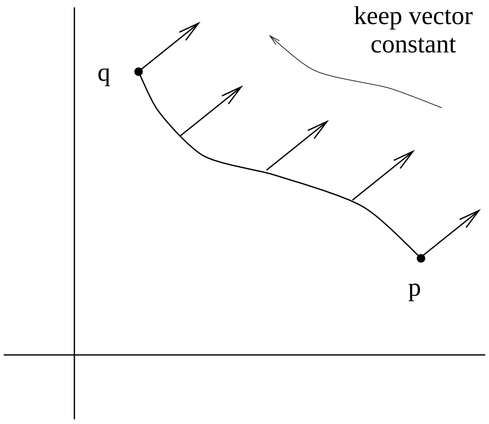
$$\begin{aligned} \nabla_{[\mu}\omega_{\nu]} &= \partial_{[\mu}\omega_{\nu]} - \Gamma^{\lambda}_{[\mu\nu]}\omega_{\lambda} \\ &= \partial_{[\mu}\omega_{\nu]} \ . \end{aligned} \tag{3.29}$$

This has led some misfortunate souls to fret about the "ambiguity" of the exterior derivative in spaces with torsion, where the above simplification does not occur. There is no ambiguity: the exterior derivative does not involve the connection, no matter what connection you happen to be using, and therefore the torsion never enters the formula for the exterior derivative of anything.

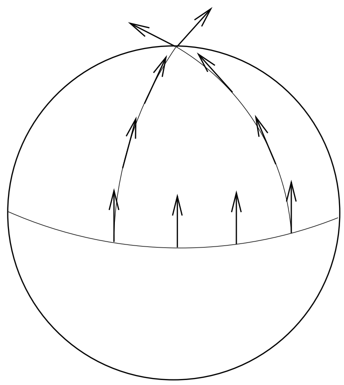
Before moving on, let's review the process by which we have been adding structures to our mathematical constructs. We started with the basic notion of a set, which you were presumed to know (informally, if not rigorously). We introduced the concept of open subsets of our set; this is equivalent to introducing a topology, and promoted the set to a topological space. Then by demanding that each open set look like a region of $\mathbf{R}^n$ (with n the same for each set) and that the coordinate charts be smoothly sewn together, the topological space became a manifold. A manifold is simultaneously a very flexible and powerful structure, and comes equipped naturally with a tangent bundle, tensor bundles of various ranks, the ability to take exterior derivatives, and so forth. We then proceeded to put a metric on the manifold, resulting in a manifold with metric (or sometimes "Riemannian manifold"). Independently of the metric we found we could introduce a connection, allowing us to take covariant derivatives. Once we have a metric, however, there is automatically a unique torsion-free metric-compatible connection. (In principle there is nothing to stop us from introducing more than one connection, or more than one metric, on any given manifold.) The situation is thus as portrayed in the diagram on the next page.



Having set up the machinery of connections, the first thing we will do is discuss parallel transport. Recall that in flat space it was unnecessary to be very careful about the fact that vectors were elements of tangent spaces defined at individual points; it is actually very natural to compare vectors at different points (where by "compare" we mean add, subtract, take the dot product, etc.). The reason why it is natural is because it makes sense, in flat space, to "move a vector from one point to another while keeping it constant." Then once we get the vector from one point to another we can do the usual operations allowed in a vector space.



The concept of moving a vector along a path, keeping constant all the while, is known as parallel transport. As we shall see, parallel transport is defined whenever we have a connection; the intuitive manipulation of vectors in flat space makes implicit use of the Christoffel connection on this space. The crucial difference between flat and curved spaces is that, in a curved space, *the result of parallel transporting a vector from one point to another will depend on the path taken between the points.* Without yet assembling the complete mechanism of parallel transport, we can use our intuition about the two-sphere to see that this is the case. Start with a vector on the equator, pointing along a line of constant longitude. Parallel transport it up to the north pole along a line of longitude in the obvious way. Then take the original vector, parallel transport it along the equator by an angle θ, and then move it up to the north pole as before. It is clear that the vector, parallel transported along two paths, arrived at the same destination with two different values (rotated by θ).



It therefore appears as if there is no natural way to uniquely move a vector from one tangent space to another; we can always parallel transport it, but the result depends on the path, and there is no natural choice of which path to take. Unlike some of the problems we have encountered, *there is no solution to this one* — we simply must learn to live with the fact that two vectors can only be compared in a natural way if they are elements of the same tangent space. For example, two particles passing by each other have a well-defined relative velocity (which cannot be greater than the speed of light). But two particles at different points on a curved manifold do not have any well-defined notion of relative velocity — the concept simply makes no sense. Of course, in certain special situations it is still useful to talk as if it did make sense, but it is necessary to understand that occasional usefulness is not a substitute for rigorous definition. In cosmology, for example, the light from distant galaxies is redshifted with respect to the frequencies we would observe from a nearby stationary source. Since this phenomenon bears such a close resemblance to the conventional Doppler effect due to relative motion, it is very tempting to say that the galaxies are "receding away from us" at a speed defined by their redshift. At a rigorous level this is nonsense, what Wittgenstein would call a "grammatical mistake" — the galaxies are not receding, since the notion of their velocity with respect to us is not well-defined. What is actually happening is that the metric of spacetime between us and the galaxies has changed (the universe has expanded) along the path of the photon from here to there, leading to an increase in the wavelength of the light. As an example of how you can go wrong, naive application of the Doppler formula to the redshift of galaxies implies that some of them are receding faster than light, in apparent contradiction with relativity. The resolution of this apparent paradox is simply that the very notion of their recession should not be taken literally.

Enough about what we cannot do; let's see what we can. Parallel transport is supposed to be the curved-space generalization of the concept of "keeping the vector constant" as we move it along a path; similarly for a tensor of arbitrary rank. Given a curve $x^\mu(\lambda)$, the requirement of constancy of a tensor T along this curve in flat space is simply $\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu} = 0$. We therefore define the covariant derivative along the path to be given by an operator

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu \ . \tag{3.30}$$

We then define **parallel transport** of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that, along the path,

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = 0 \ . \tag{3.31}$$

This is a well-defined tensor equation, since both the tangent vector $dx^\mu/d\lambda$ and the covariant derivative ∇T are tensors. This is known as the **equation of parallel transport**. For a vector it takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0 \ . \tag{3.32}$$

We can look at the parallel transport equation as a first-order differential equation defining an initial-value problem: given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path such that the continuation solves (3.31). We say that such a tensor is parallel transported.

The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it:

$$\frac{D}{d\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0 \ . \tag{3.33}$$

It follows that the inner product of two parallel-transported vectors is preserved. That is, if V^μ and W^ν are parallel-transported along a curve $x^\sigma(\lambda)$, we have

$$\begin{aligned}\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) &= \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right) \\ &= 0 \ .\end{aligned} \tag{3.34}$$

This means that parallel transport with respect to a metric-compatible connection preserves the norm of vectors, the sense of orthogonality, and so on.

One thing they don't usually tell you in GR books is that you can write down an explicit and general solution to the parallel transport equation, although it's somewhat formal. First notice that for some path $\gamma : \lambda \to x^\sigma(\lambda)$, solving the parallel transport equation for a vector V^μ amounts to finding a matrix $P^\mu{}_\rho(\lambda, \lambda_0)$ which relates the vector at its initial value $V^\mu(\lambda_0)$ to its value somewhere later down the path:

$$V^\mu(\lambda) = P^\mu{}_\rho(\lambda, \lambda_0) V^\rho(\lambda_0) . \tag{3.35}$$

Of course the matrix $P^\mu{}_\rho(\lambda, \lambda_0)$, known as the **parallel propagator**, depends on the path γ (although it's hard to find a notation which indicates this without making γ look like an index). If we define

$$A^\mu{}_\rho(\lambda) = -\Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} , \tag{3.36}$$

where the quantities on the right hand side are evaluated at $x^\nu(\lambda)$, then the parallel transport equation becomes

$$\frac{d}{d\lambda} V^\mu = A^\mu{}_\rho V^\rho . \tag{3.37}$$

Since the parallel propagator must work for any vector, substituting (3.35) into (3.37) shows that $P^\mu{}_\rho(\lambda, \lambda_0)$ also obeys this equation:

$$\frac{d}{d\lambda} P^\mu{}_\rho(\lambda, \lambda_0) = A^\mu{}_\sigma(\lambda) P^\sigma{}_\rho(\lambda, \lambda_0) . \tag{3.38}$$

To solve this equation, first integrate both sides:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} A^\mu{}_\sigma(\eta) P^\sigma{}_\rho(\eta, \lambda_0)\, d\eta . \tag{3.39}$$

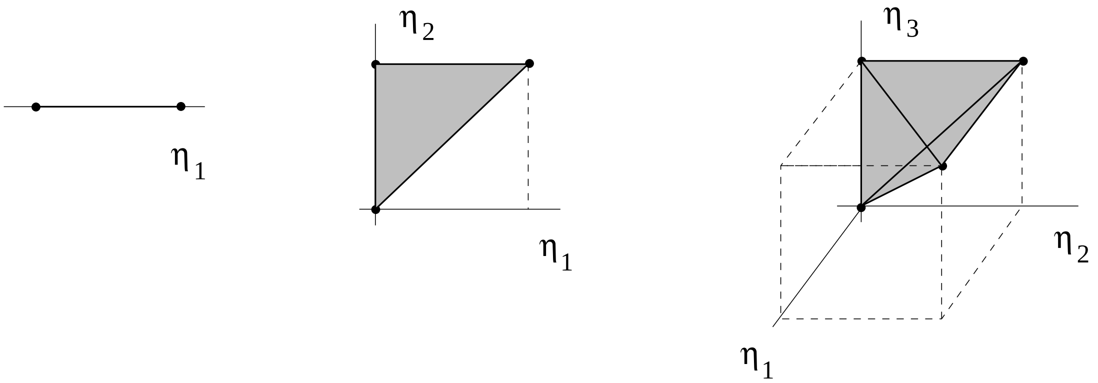
The Kronecker delta, it is easy to see, provides the correct normalization for $\lambda = \lambda_0$.

We can solve (3.39) by iteration, taking the right hand side and plugging it into itself repeatedly, giving

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} A^\mu{}_\rho(\eta)\, d\eta + \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\eta} A^\mu{}_\sigma(\eta) A^\sigma{}_\rho(\eta')\, d\eta' d\eta + \cdots . \tag{3.40}$$

The nth term in this series is an integral over an n-dimensional right triangle, or n-simplex.

$$\int_{\lambda_0}^{\lambda} A(\eta_1)\, d\eta_1 \qquad \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_2} A(\eta_2)A(\eta_1)\, d\eta_1 d\eta_2 \qquad \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_3}\int_{\lambda_0}^{\eta_2} A(\eta_3)A(\eta_2)A(\eta_1)\, d^3\eta$$



It would simplify things if we could consider such an integral to be over an n-cube instead of an n-simplex; is there some way to do this? There are $n!$ such simplices in each cube, so we would have to multiply by $1/n!$ to compensate for this extra volume. But we also want to get the integrand right; using matrix notation, the integrand at nth order is $A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)$, but with the special property that $\eta_n \geq \eta_{n-1} \geq \cdots \geq \eta_1$. We therefore define the **path-ordering symbol**, $\mathcal{P}$, to ensure that this condition holds. In other words, the expression

$$\mathcal{P}[A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)] \tag{3.41}$$

stands for the product of the n matrices $A(\eta_i)$, ordered in such a way that the largest value of η_i is on the left, and each subsequent value of η_i is less than or equal to the previous one. We then can express the nth-order term in (3.40) as

$$\begin{aligned} &\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_n}\cdots\int_{\lambda_0}^{\eta_2} A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)\, d^n\eta \\ &= \frac{1}{n!}\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda}\cdots\int_{\lambda_0}^{\lambda}\mathcal{P}[A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)]\, d^n\eta\ . \end{aligned} \tag{3.42}$$

This expression contains no substantive statement about the matrices $A(\eta_i)$; it is just notation. But we can now write (3.40) in matrix form as

$$P(\lambda,\lambda_0) = \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\lambda_0}^{\lambda}\mathcal{P}[A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)]\, d^n\eta\ . \tag{3.43}$$

This formula is just the series expression for an exponential; we therefore say that the parallel propagator is given by the path-ordered exponential

$$P(\lambda,\lambda_0) = \mathcal{P}\exp\left(\int_{\lambda_0}^{\lambda} A(\eta)\, d\eta\right)\ , \tag{3.44}$$

where once again this is just notation; the path-ordered exponential is defined to be the right hand side of (3.43). We can write it more explicitly as

$$P^{\mu}{}_{\nu}(\lambda, \lambda_0) = \mathcal{P} \exp\left(-\int_{\lambda_0}^{\lambda} \Gamma^{\mu}_{\sigma\nu} \frac{dx^{\sigma}}{d\eta}\, d\eta \right) . \tag{3.45}$$

It's nice to have an explicit formula, even if it is rather abstract. The same kind of expression appears in quantum field theory as "Dyson's Formula," where it arises because the Schrödinger equation for the time-evolution operator has the same form as (3.38).

As an aside, an especially interesting example of the parallel propagator occurs when the path is a loop, starting and ending at the same point. Then if the connection is metric-compatible, the resulting matrix will just be a Lorentz transformation on the tangent space at the point. This transformation is known as the "holonomy" of the loop. If you know the holonomy of every possible loop, that turns out to be equivalent to knowing the metric. This fact has let Ashtekar and his collaborators to examine general relativity in the "loop representation," where the fundamental variables are holonomies rather than the explicit metric. They have made some progress towards quantizing the theory in this approach, although the jury is still out about how much further progress can be made.

With parallel transport understood, the next logical step is to discuss geodesics. A geodesic is the curved-space generalization of the notion of a "straight line" in Euclidean space. We all know what a straight line is: it's the path of shortest distance between two points. But there is an equally good definition — a straight line is a path which parallel transports its own tangent vector. On a manifold with an arbitrary (not necessarily Christoffel) connection, these two concepts do not quite coincide, and we should discuss them separately.

We'll take the second definition first, since it is computationally much more straightforward. The tangent vector to a path $x^{\mu}(\lambda)$ is $dx^{\mu}/d\lambda$. The condition that it be parallel transported is thus

$$\frac{D}{d\lambda}\frac{dx^{\mu}}{d\lambda} = 0 , \tag{3.46}$$

or alternatively

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0 . \tag{3.47}$$

This is the **geodesic equation**, another one which you should memorize. We can easily see that it reproduces the usual notion of straight lines if the connection coefficients are the Christoffel symbols in Euclidean space; in that case we can choose Cartesian coordinates in which $\Gamma^{\mu}_{\rho\sigma} = 0$, and the geodesic equation is just $d^2x^{\mu}/d\lambda^2 = 0$, which is the equation for a straight line.

That was embarrassingly simple; let's turn to the more nontrivial case of the shortest distance definition. As we know, there are various subtleties involved in the definition of distance in a Lorentzian spacetime; for null paths the distance is zero, for timelike paths it's more convenient to use the proper time, etc. So in the name of simplicity let's do the calculation just for a timelike path — the resulting equation will turn out to be good for any path, so we are not losing any generality. We therefore consider the proper time functional,

$$\tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda \ , \tag{3.48}$$

where the integral is over the path. To search for shortest-distance paths, we will do the usual calculus of variations treatment to seek extrema of this functional. (In fact they will turn out to be curves of *maximum* proper time.)

We want to consider the change in the proper time under infinitesimal variations of the path,

$$\begin{aligned} x^\mu &\rightarrow x^\mu + \delta x^\mu \\ g_{\mu\nu} &\rightarrow g_{\mu\nu} + \delta x^\sigma \partial_\sigma g_{\mu\nu} \ . \end{aligned} \tag{3.49}$$

(The second line comes from Taylor expansion in curved spacetime, which as you can see uses the partial derivative, not the covariant derivative.) Plugging this into (3.48), we get

$$\begin{aligned} \tau + \delta\tau &= \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\sigma - 2g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d(\delta x^\nu)}{d\lambda}\right)^{1/2} d\lambda \\ &= \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} \left[1 + \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1} \right. \\ &\qquad \left. \times \left(-\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\sigma - 2g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d(\delta x^\nu)}{d\lambda}\right)\right]^{1/2} d\lambda \ . \end{aligned} \tag{3.50}$$

Since δx^σ is assumed to be small, we can expand the square root of the expression in square brackets to find

$$\delta\tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2} \left(-\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\sigma - g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d(\delta x^\nu)}{d\lambda}\right) d\lambda \ . \tag{3.51}$$

It is helpful at this point to change the parameterization of our curve from λ, which was arbitrary, to the proper time τ itself, using

$$d\lambda = \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2} d\tau \ . \tag{3.52}$$

We plug this into (3.51) (note: we plug it in for every appearance of $d\lambda$) to obtain

$$\delta\tau = \int \left[-\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma - g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}\right] d\tau$$

$$= \int \left[-\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d}{d\tau}\left(g_{\mu\sigma}\frac{dx^\mu}{d\tau}\right)\right]\delta x^\sigma \, d\tau \ , \tag{3.53}$$

where in the last line we have integrated by parts, avoiding possible boundary contributions by demanding that the variation δx^σ vanish at the endpoints of the path. Since we are searching for stationary points, we want $\delta\tau$ to vanish for any variation; this implies

$$-\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\nu g_{\mu\sigma} + g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} = 0 \ , \tag{3.54}$$

where we have used $dg_{\mu\sigma}/d\tau = (dx^\nu/d\tau)\partial_\nu g_{\mu\sigma}$. Some shuffling of dummy indices reveals

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(-\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \ , \tag{3.55}$$

and multiplying by the inverse metric finally leads to

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \ . \tag{3.56}$$

We see that this is precisely the geodesic equation (3.32), but with the specific choice of Christoffel connection (3.21). Thus, on a manifold with metric, extremals of the length functional are curves which parallel transport their tangent vector with respect to the Christoffel connection associated with that metric. It doesn't matter if there is any other connection defined on the same manifold. Of course, in GR the Christoffel connection is the only one which is used, so the two notions are the same.

The primary usefulness of geodesics in general relativity is that they are the paths followed by unaccelerated particles. In fact, the geodesic equation can be thought of as the generalization of Newton's law $\mathbf{f} = m\mathbf{a}$ for the case $\mathbf{f} = 0$. It is also possible to introduce forces by adding terms to the right hand side; in fact, looking back to the expression (1.103) for the Lorentz force in special relativity, it is tempting to guess that the equation of motion for a particle of mass m and charge q in general relativity should be

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau} \ . \tag{3.57}$$

We will talk about this more later, but in fact your guess would be correct.

Having boldly derived these expressions, we should say some more careful words about the parameterization of a geodesic path. When we presented the geodesic equation as the requirement that the tangent vector be parallel transported, (3.47), we parameterized our path with some parameter λ, whereas when we found the formula (3.56) for the extremal of the spacetime interval we wound up with a very specific parameterization, the proper time. Of course from the form of (3.56) it is clear that a transformation

$$\tau \to \lambda = a\tau + b \ , \tag{3.58}$$

for some constants a and b, leaves the equation invariant. Any parameter related to the proper time in this way is called an **affine parameter**, and is just as good as the proper time for parameterizing a geodesic. What was hidden in our derivation of (3.47) was that *the demand that the tangent vector be parallel transported actually constrains the parameterization of the curve*, specifically to one related to the proper time by (3.58). In other words, if you start at some point and with some initial direction, and then construct a curve by beginning to walk in that direction and keeping your tangent vector parallel transported, you will not only define a path in the manifold but also (up to linear transformations) define the parameter along the path.

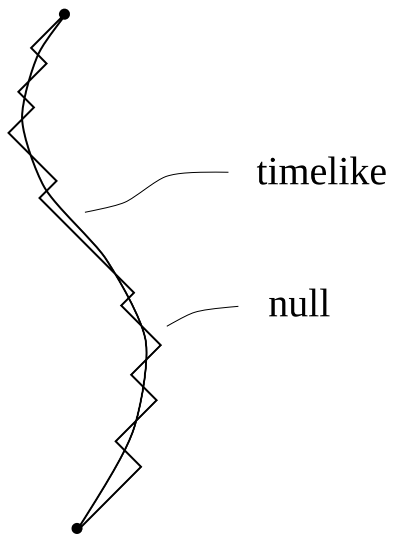
Of course, there is nothing to stop you from using any other parameterization you like, but then (3.47) will not be satisfied. More generally you will satisfy an equation of the form

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha} = f(\alpha)\frac{dx^\mu}{d\alpha} \ , \tag{3.59}$$

for some parameter α and some function $f(\alpha)$. Conversely, if (3.59) is satisfied along a curve you can always find an affine parameter $\lambda(\alpha)$ for which the geodesic equation (3.47) will be satisfied.

An important property of geodesics in a spacetime with Lorentzian metric is that the character (timelike/null/spacelike) of the geodesic (relative to a metric-compatible connection) never changes. This is simply because parallel transport preserves inner products, and the character is determined by the inner product of the tangent vector with itself. This is why we were consistent to consider purely timelike paths when we derived (3.56); for spacelike paths we would have derived the same equation, since the only difference is an overall minus sign in the final answer. There are also null geodesics, which satisfy the same equation, except that the proper time cannot be used as a parameter (some set of allowed parameters will exist, related to each other by linear transformations). You can derive this fact either from the simple requirement that the tangent vector be parallel transported, or by extending the variation of (3.48) to include all non-spacelike paths.

Let's now explain the earlier remark that timelike geodesics are maxima of the proper time. The reason we know this is true is that, given any timelike curve (geodesic or not), we can approximate it to arbitrary accuracy by a null curve. To do this all we have to do is to consider "jagged" null curves which follow the timelike one:



As we increase the number of sharp corners, the null curve comes closer and closer to the timelike curve while still having zero path length. Timelike geodesics cannot therefore be curves of minimum proper time, since they are always infinitesimally close to curves of zero proper time; in fact they maximize the proper time. (This is how you can remember which twin in the twin paradox ages more — the one who stays home is basically on a geodesic, and therefore experiences more proper time.) Of course even this is being a little cavalier; actually every time we say "maximize" or "minimize" we should add the modifier "locally." It is often the case that between two points on a manifold there is more than one geodesic. For instance, on S^2 we can draw a great circle through any two points, and imagine travelling between them either the short way or the long way around. One of these is obviously longer than the other, although both are stationary points of the length functional.

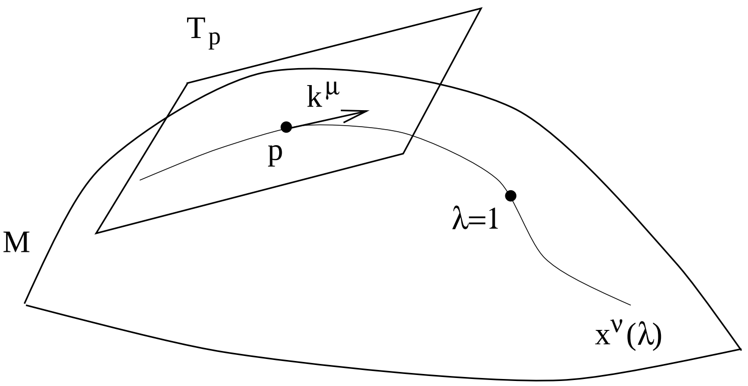
The final fact about geodesics before we move on to curvature proper is their use in mapping the tangent space at a point p to a local neighborhood of p. To do this we notice that any geodesic $x^\mu(\lambda)$ which passes through p can be specified by its behavior at p; let us choose the parameter value to be $\lambda(p) = 0$, and the tangent vector at p to be

$$\frac{dx^\mu}{d\lambda}(\lambda = 0) = k^\mu \ , \tag{3.60}$$

for k^μ some vector at p (some element of T_p). Then there will be a unique point on the manifold M which lies on this geodesic where the parameter has the value $\lambda = 1$. We define the **exponential map** at p, $\exp_p : T_p \to M$, via

$$\exp_p(k^\mu) = x^\nu(\lambda = 1) \ , \tag{3.61}$$

where $x^\nu(\lambda)$ solves the geodesic equation subject to (3.60). For some set of tangent vectors k^μ near the zero vector, this map will be well-defined, and in fact invertible. Thus in the



neighborhood of p given by the range of the map on this set of tangent vectors, the the tangent vectors themselves define a coordinate system on the manifold. In this coordinate system, any geodesic through p is expressed trivially as

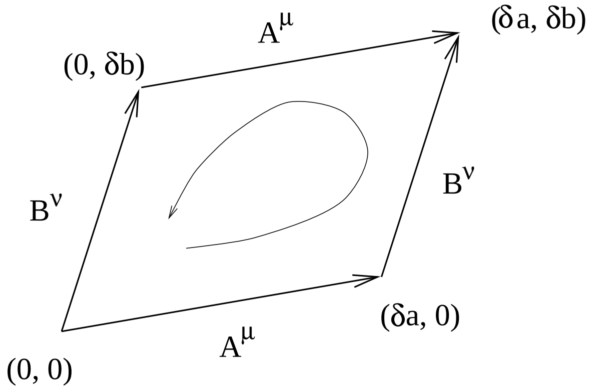
$$x^\mu(\lambda) = \lambda k^\mu \ , \tag{3.62}$$

for some appropriate vector k^μ.

We won't go into detail about the properties of the exponential map, since in fact we won't be using it much, but it's important to emphasize that the range of the map is not necessarily the whole manifold, and the domain is not necessarily the whole tangent space. The range can fail to be all of M simply because there can be two points which are not connected by any geodesic. (In a Euclidean signature metric this is impossible, but not in a Lorentzian spacetime.) The domain can fail to be all of T_p because a geodesic may run into a singularity, which we think of as "the edge of the manifold." Manifolds which have such singularities are known as **geodesically incomplete**. This is not merely a problem for careful mathematicians; in fact the "singularity theorems" of Hawking and Penrose state that, for reasonable matter content (no negative energies), spacetimes in general relativity are almost guaranteed to be geodesically incomplete. As examples, the two most useful spacetimes in GR — the Schwarzschild solution describing black holes and the Friedmann-Robertson-Walker solutions describing homogeneous, isotropic cosmologies — both feature important singularities.

Having set up the machinery of parallel transport and covariant derivatives, we are at last prepared to discuss curvature proper. The curvature is quantified by the Riemann tensor, which is derived from the connection. The idea behind this measure of curvature is that we know what we mean by "flatness" of a connection — the conventional (and usually implicit) Christoffel connection associated with a Euclidean or Minkowskian metric has a number of properties which can be thought of as different manifestations of flatness. These include the fact that parallel transport around a closed loop leaves a vector unchanged, that covariant derivatives of tensors commute, and that initially parallel geodesics remain parallel. As we shall see, the Riemann tensor arises when we study how any of these properties are altered in more general contexts.

We have already argued, using the two-sphere as an example, that parallel transport of a vector around a closed loop in a curved space will lead to a transformation of the vector. The resulting transformation depends on the total curvature enclosed by the loop; it would be more useful to have a local description of the curvature at each point, which is what the Riemann tensor is supposed to provide. One conventional way to introduce the Riemann tensor, therefore, is to consider parallel transport around an infinitesimal loop. We are not going to do that here, but take a more direct route. (Most of the presentations in the literature are either sloppy, or correct but very difficult to follow.) Nevertheless, even without working through the details, it is possible to see what form the answer should take. Imagine that we parallel transport a vector V^σ around a closed loop defined by two vectors A^ν and B^μ:



The (infinitesimal) lengths of the sides of the loop are δa and δb, respectively. Now, we know the action of parallel transport is independent of coordinates, so there should be some tensor which tells us how the vector changes when it comes back to its starting point; it will be a linear transformation on a vector, and therefore involve one upper and one lower index. But it will also depend on the two vectors A and B which define the loop; therefore there should be two additional lower indices to contract with A^ν and B^μ. Furthermore, the tensor should be antisymmetric in these two indices, since interchanging the vectors corresponds to traversing the loop in the opposite direction, and should give the inverse of the original answer. (This is consistent with the fact that the transformation should vanish if A and B are the same vector.) We therefore expect that the expression for the change δV^ρ experienced by this vector when parallel transported around the loop should be of the form

$$\delta V^\rho = (\delta a)(\delta b) A^\nu B^\mu R^\rho{}_{\sigma\mu\nu} V^\sigma \ , \tag{3.63}$$

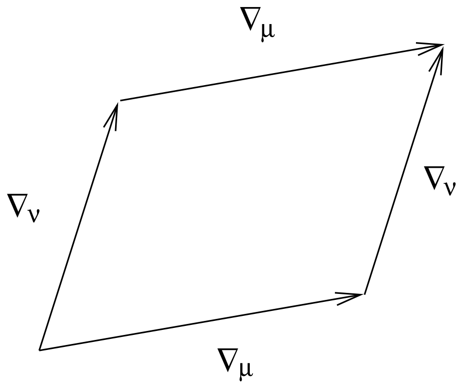
where $R^\rho{}_{\sigma\mu\nu}$ is a $(1,3)$ tensor known as the **Riemann tensor** (or simply "curvature tensor").

It is antisymmetric in the last two indices:

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \ . \tag{3.64}$$

(Of course, if (3.63) is taken as a definition of the Riemann tensor, there is a convention that needs to be chosen for the ordering of the indices. There is no agreement at all on what this convention should be, so be careful.)

Knowing what we do about parallel transport, we could very carefully perform the necessary manipulations to see what happens to the vector under this operation, and the result would be a formula for the curvature tensor in terms of the connection coefficients. It is much quicker, however, to consider a related operation, the commutator of two covariant derivatives. The relationship between this and parallel transport around a loop should be evident; the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported (since the covariant derivative of a tensor in a direction along which it is parallel transported is zero). The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first one way and then the other, versus the opposite ordering.



The actual computation is very straightforward. Considering a vector field V^ρ, we take

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\nu\sigma})V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}V^\sigma \\
&\qquad + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho \ .
\end{aligned} \tag{3.65}$$

In the last step we have relabeled some dummy indices and eliminated some terms that cancel when antisymmetrized. We recognize that the last term is simply the torsion tensor, and that the left hand side is manifestly a tensor; therefore the expression in parentheses must be a tensor itself. We write

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T_{\mu\nu}{}^\lambda\nabla_\lambda V^\rho \ , \tag{3.66}$$

where the Riemann tensor is identified as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \ . \tag{3.67}$$

There are a number of things to notice about the derivation of this expression:

- Of course we have not demonstrated that (3.67) is actually the same tensor that appeared in (3.63), but in fact it's true (see Wald for a believable if tortuous demonstration).

- It is perhaps surprising that the commutator $[\nabla_\mu, \nabla_\nu]$, which appears to be a differential operator, has an action on vector fields which (in the absence of torsion, at any rate) is a simple multiplicative transformation. The Riemann tensor measures that part of the commutator of covariant derivatives which is proportional to the vector field, while the torsion tensor measures the part which is proportional to the covariant derivative of the vector field; the second derivative doesn't enter at all.

- Notice that the expression (3.67) is constructed from non-tensorial elements; you can check that the transformation laws all work out to make this particular combination a legitimate tensor.

- The antisymmetry of $R^\rho{}_{\sigma\mu\nu}$ in its last two indices is immediate from this formula and its derivation.

- We constructed the curvature tensor completely from the connection (no mention of the metric was made). We were sufficiently careful that the above expression is true for any connection, whether or not it is metric compatible or torsion free.

- Using what are by now our usual methods, the action of $[\nabla_\rho, \nabla_\sigma]$ can be computed on a tensor of arbitrary rank. The answer is

$$\begin{aligned}[\nabla_\rho, \nabla_\sigma] X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = & - T_{\rho\sigma}{}^\lambda \nabla_\lambda X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \\ & + R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots \\ & - R^\lambda{}_{\nu_1\rho\sigma} X^{\mu_1\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - R^\lambda{}_{\nu_2\rho\sigma} X^{\mu_1\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots \end{aligned} \tag{3.68}$$

A useful notion is that of the commutator of two vector fields X and Y, which is a third vector field with components

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu \ . \tag{3.69}$$

Both the torsion tensor and the Riemann tensor, thought of as multilinear maps, have elegant expressions in terms of the commutator. Thinking of the torsion as a map from two vector fields to a third vector field, we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \ , \tag{3.70}$$

and thinking of the Riemann tensor as a map from three vector fields to a fourth one, we have

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ . \tag{3.71}$$

In these expressions, the notation ∇_X refers to the covariant derivative along the vector field X; in components, $\nabla_X = X^\mu\nabla_\mu$. Note that the two vectors X and Y in (3.71) correspond to the two antisymmetric indices in the component form of the Riemann tensor. The last term in (3.71), involving the commutator $[X,Y]$, vanishes when X and Y are taken to be the coordinate basis vector fields (since $[\partial_\mu, \partial_\nu] = 0$), which is why this term did not arise when we originally took the commutator of two covariant derivatives. We will not use this notation extensively, but you might see it in the literature, so you should be able to decode it.

Having defined the curvature tensor as something which characterizes the connection, let us now admit that in GR we are most concerned with the Christoffel connection. In this case the connection is derived from the metric, and the associated curvature may be thought of as that of the metric itself. This identification allows us to finally make sense of our informal notion that spaces for which the metric looks Euclidean or Minkowskian are flat. In fact it works both ways: if the components of the metric are constant in some coordinate system, the Riemann tensor will vanish, while if the Riemann tensor vanishes we can always construct a coordinate system in which the metric components are constant.

The first of these is easy to show. If we are in some coordinate system such that $\partial_\sigma g_{\mu\nu} = 0$ (everywhere, not just at a point), then $\Gamma^\rho_{\mu\nu} = 0$ and $\partial_\sigma\Gamma^\rho_{\mu\nu} = 0$; thus $R^\rho{}_{\sigma\mu\nu} = 0$ by (3.67). But this is a tensor equation, and if it is true in one coordinate system it must be true in any coordinate system. Therefore, the statement that the Riemann tensor vanishes is a necessary condition for it to be possible to find coordinates in which the components of $g_{\mu\nu}$ are constant everywhere.

It is also a sufficient condition, although we have to work harder to show it. Start by choosing Riemann normal coordinates at some point p, so that $g_{\mu\nu} = \eta_{\mu\nu}$ at p. (Here we are using $\eta_{\mu\nu}$ in a generalized sense, as a matrix with either $+1$ or -1 for each diagonal element and zeroes elsewhere. The actual arrangement of the $+1$'s and -1's depends on the canonical form of the metric, but is irrelevant for the present argument.) Denote the basis vectors at p by $\hat{e}_{(\mu)}$, with components $\hat{e}^\sigma_{(\mu)}$. Then by construction we have

$$g_{\sigma\rho}\hat{e}^\sigma_{(\mu)}\hat{e}^\rho_{(\nu)}(p) = \eta_{\mu\nu} \ . \tag{3.72}$$

Now let us parallel transport the entire set of basis vectors from p to another point q; the vanishing of the Riemann tensor ensures that the result will be independent of the path taken between p and q. Since parallel transport with respect to a metric compatible connection preserves inner products, we must have

$$g_{\sigma\rho}\hat{e}^\sigma_{(\mu)}\hat{e}^\rho_{(\nu)}(q) = \eta_{\mu\nu} \ . \tag{3.73}$$

We therefore have specified a set of vector fields which everywhere define a basis in which the metric components are constant. This is completely unimpressive; it can be done on any manifold, regardless of what the curvature is. What we would like to show is that this is a coordinate basis (which can only be true if the curvature vanishes). We know that if the $\hat{e}_{(\mu)}$'s are a coordinate basis, their commutator will vanish:

$$[\hat{e}_{(\mu)}, \hat{e}_{(\nu)}] = 0 \ . \tag{3.74}$$

What we would really like is the converse: that if the commutator vanishes we can find coordinates y^μ such that $\hat{e}_{(\mu)} = \frac{\partial}{\partial y^\mu}$. In fact this is a true result, known as **Frobenius's Theorem**. It's something of a mess to prove, involving a good deal more mathematical apparatus than we have bothered to set up. Let's just take it for granted (skeptics can consult Schutz's *Geometrical Methods* book). Thus, we would like to demonstrate (3.74) for the vector fields we have set up. Let's use the expression (3.70) for the torsion:

$$[\hat{e}_{(\mu)}, \hat{e}_{(\nu)}] = \nabla_{\hat{e}_{(\mu)}}\hat{e}_{(\nu)} - \nabla_{\hat{e}_{(\nu)}}\hat{e}_{(\mu)} - T(\hat{e}_{(\mu)}, \hat{e}_{(\nu)}) \ . \tag{3.75}$$

The torsion vanishes by hypothesis. The covariant derivatives will also vanish, given the method by which we constructed our vector fields; they were made by parallel transporting along arbitrary paths. If the fields are parallel transported along arbitrary paths, they are certainly parallel transported along the vectors $\hat{e}_{(\mu)}$, and therefore their covariant derivatives in the direction of these vectors will vanish. Thus (3.70) implies that the commutator vanishes, and therefore that we can find a coordinate system y^μ for which these vector fields are the partial derivatives. In this coordinate system the metric will have components $\eta_{\mu\nu}$, as desired.

The Riemann tensor, with four indices, naively has n^4 independent components in an n-dimensional space. In fact the antisymmetry property (3.64) means that there are only $n(n-1)/2$ independent values these last two indices can take on, leaving us with $n^3(n-1)/2$ independent components. When we consider the Christoffel connection, however, there are a number of other symmetries that reduce the independent components further. Let's consider these now.

The simplest way to derive these additional symmetries is to examine the Riemann tensor with all lower indices,

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^\lambda{}_{\sigma\mu\nu} \ . \tag{3.76}$$

Let us further consider the components of this tensor in Riemann normal coordinates established at a point p. Then the Christoffel symbols themselves will vanish, although their derivatives will not. We therefore have

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}(\partial_\mu\Gamma^\lambda_{\nu\sigma} - \partial_\nu\Gamma^\lambda_{\mu\sigma})$$

$$
\begin{aligned}
&= \frac{1}{2} g_{\rho\lambda} g^{\lambda\tau} (\partial_\mu \partial_\nu g_{\sigma\tau} + \partial_\mu \partial_\sigma g_{\tau\nu} - \partial_\mu \partial_\tau g_{\nu\sigma} - \partial_\nu \partial_\mu g_{\sigma\tau} - \partial_\nu \partial_\sigma g_{\tau\mu} + \partial_\nu \partial_\tau g_{\mu\sigma}) \\
&= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\nu \partial_\rho g_{\mu\sigma}) \ . \qquad (3.77)
\end{aligned}
$$

In the second line we have used $\partial_\mu g^{\lambda\tau} = 0$ in RNC's, and in the third line the fact that partials commute. From this expression we can notice immediately two properties of $R_{\rho\sigma\mu\nu}$; it is antisymmetric in its first two indices,

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \ , \qquad (3.78)$$

and it is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \ . \qquad (3.79)$$

With a little more work, which we leave to your imagination, we can see that the sum of cyclic permutations of the last three indices vanishes:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \ . \qquad (3.80)$$

This last property is equivalent to the vanishing of the antisymmetric part of the last three indices:

$$R_{\rho[\sigma\mu\nu]} = 0 \ . \qquad (3.81)$$

All of these properties have been derived in a special coordinate system, but they are all tensor equations; therefore they will be true in any coordinates. Not all of them are independent; with some effort, you can show that (3.64), (3.78) and (3.81) together imply (3.79). The logical interdependence of the equations is usually less important than the simple fact that they are true.

Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Let's begin with the facts that $R_{\rho\sigma\mu\nu}$ is antisymmetric in the first two indices, antisymmetric in the last two indices, and symmetric under interchange of these two pairs. This means that we can think of it as a symmetric matrix $R_{[\rho\sigma][\mu\nu]}$, where the pairs $\rho\sigma$ and $\mu\nu$ are thought of as individual indices. An $m \times m$ symmetric matrix has $m(m+1)/2$ independent components, while an $n \times n$ antisymmetric matrix has $n(n-1)/2$ independent components. We therefore have

$$\frac{1}{2}\left[\frac{1}{2}n(n-1)\right]\left[\frac{1}{2}n(n-1)+1\right] = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) \qquad (3.82)$$

independent components. We still have to deal with the additional symmetry (3.81). An immediate consequence of (3.81) is that the totally antisymmetric part of the Riemann tensor vanishes,

$$R_{[\rho\sigma\mu\nu]} = 0 \ . \qquad (3.83)$$

In fact, this equation plus the other symmetries (3.64), (3.78) and (3.79) are enough to imply (3.81), as can be easily shown by expanding (3.83) and messing with the resulting terms. Therefore imposing the additional constraint of (3.83) is equivalent to imposing (3.81), once the other symmetries have been accounted for. How many independent restrictions does this represent? Let us imagine decomposing

$$R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{[\rho\sigma\mu\nu]} \ . \tag{3.84}$$

It is easy to see that any totally antisymmetric 4-index tensor is automatically antisymmetric in its first and last indices, and symmetric under interchange of the two pairs. Therefore these properties are independent restrictions on $X_{\rho\sigma\mu\nu}$, unrelated to the requirement (3.83). Now a totally antisymmetric 4-index tensor has $n(n-1)(n-2)(n-3)/4!$ terms, and therefore (3.83) reduces the number of independent components by this amount. We are left with

$$\frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1) \tag{3.85}$$

independent components of the Riemann tensor.

In four dimensions, therefore, the Riemann tensor has 20 independent components. (In one dimension it has none.) These twenty functions are precisely the 20 degrees of freedom in the second derivatives of the metric which we could not set to zero by a clever choice of coordinates. This should reinforce your confidence that the Riemann tensor is an appropriate measure of curvature.

In addition to the algebraic symmetries of the Riemann tensor (which constrain the number of independent components at any point), there is a differential identity which it obeys (which constrains its relative values at different points). Consider the covariant derivative of the Riemann tensor, evaluated in Riemann normal coordinates:

$$\begin{aligned} \nabla_\lambda R_{\rho\sigma\mu\nu} &= \partial_\lambda R_{\rho\sigma\mu\nu} \\ &= \frac{1}{2}\partial_\lambda(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma}) \ . \end{aligned} \tag{3.86}$$

We would like to consider the sum of cyclic permutations of the first three indices:

$$\begin{aligned} &\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \\ &= \frac{1}{2}(\partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\lambda\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\lambda\partial_\nu\partial_\rho g_{\mu\sigma} \\ &\quad +\partial_\rho\partial_\mu\partial_\lambda g_{\sigma\nu} - \partial_\rho\partial_\mu\partial_\sigma g_{\nu\lambda} - \partial_\rho\partial_\nu\partial_\lambda g_{\sigma\mu} + \partial_\rho\partial_\nu\partial_\sigma g_{\mu\lambda} \\ &\quad +\partial_\sigma\partial_\mu\partial_\rho g_{\lambda\nu} - \partial_\sigma\partial_\mu\partial_\lambda g_{\nu\rho} - \partial_\sigma\partial_\nu\partial_\rho g_{\lambda\mu} + \partial_\sigma\partial_\nu\partial_\lambda g_{\mu\rho}) \\ &= 0 \ . \end{aligned} \tag{3.87}$$

Once again, since this is an equation between tensors it is true in any coordinate system, even though we derived it in a particular one. We recognize by now that the antisymmetry

$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ allows us to write this result as

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \ . \tag{3.88}$$

This is known as the **Bianchi identity**. (Notice that for a general connection there would be additional terms involving the torsion tensor.) It is closely related to the Jacobi identity, since (as you can show) it basically expresses

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0 \ . \tag{3.89}$$

It is frequently useful to consider contractions of the Riemann tensor. Even without the metric, we can form a contraction known as the **Ricci tensor**:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \ . \tag{3.90}$$

Notice that, for the curvature tensor formed from an arbitrary (not necessarily Christoffel) connection, there are a number of independent contractions to take. Our primary concern is with the Christoffel connection, for which (3.90) is the only independent contraction (modulo conventions for the sign, which of course change from place to place). The Ricci tensor associated with the Christoffel connection is symmetric,

$$R_{\mu\nu} = R_{\nu\mu} \ , \tag{3.91}$$

as a consequence of the various symmetries of the Riemann tensor. Using the metric, we can take a further contraction to form the **Ricci scalar**:

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} \ . \tag{3.92}$$

An especially useful form of the Bianchi identity comes from contracting twice on (3.87):

$$\begin{aligned} 0 &= g^{\nu\sigma} g^{\mu\lambda} (\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}) \\ &= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu} \ , \end{aligned} \tag{3.93}$$

or

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R \ . \tag{3.94}$$

(Notice that, unlike the partial derivative, it makes sense to raise an index on the covariant derivative, due to metric compatibility.) If we define the **Einstein tensor** as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ , \tag{3.95}$$

then we see that the twice-contracted Bianchi identity (3.94) is equivalent to

$$\nabla^\mu G_{\mu\nu} = 0 \ . \tag{3.96}$$

The Einstein tensor, which is symmetric due to the symmetry of the Ricci tensor and the metric, will be of great importance in general relativity.

The Ricci tensor and the Ricci scalar contain information about "traces" of the Riemann tensor. It is sometimes useful to consider separately those pieces of the Riemann tensor which the Ricci tensor doesn't tell us about. We therefore invent the **Weyl tensor**, which is basically the Riemann tensor with all of its contractions removed. It is given in n dimensions by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)}Rg_{\rho[\mu}g_{\nu]\sigma} \ . \tag{3.97}$$

This messy formula is designed so that all possible contractions of $C_{\rho\sigma\mu\nu}$ vanish, while it retains the symmetries of the Riemann tensor:

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= C_{[\rho\sigma][\mu\nu]} \ , \\ C_{\rho\sigma\mu\nu} &= C_{\mu\nu\rho\sigma} \ , \\ C_{\rho[\sigma\mu\nu]} &= 0 \ . \end{aligned} \tag{3.98}$$

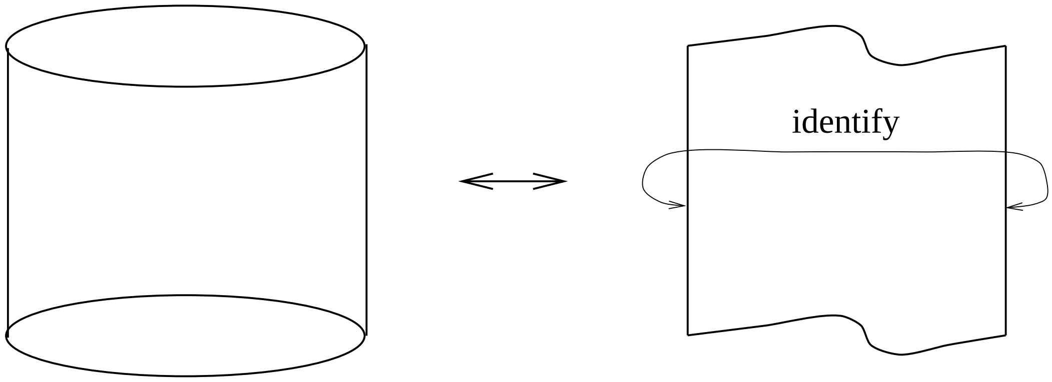
The Weyl tensor is only defined in three or more dimensions, and in three dimensions it vanishes identically. For $n \geq 4$ it satisfies a version of the Bianchi identity,

$$\nabla^\rho C_{\rho\sigma\mu\nu} = -2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(n-1)}g_{\sigma[\nu}\nabla_{\mu]}R\right) \ . \tag{3.99}$$

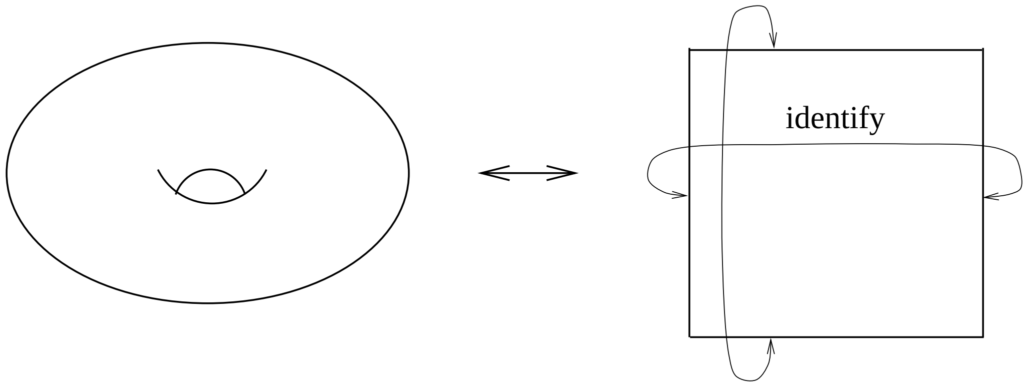
One of the most important properties of the Weyl tensor is that it is invariant under **conformal transformations**. This means that if you compute $C_{\rho\sigma\mu\nu}$ for some metric $g_{\mu\nu}$, and then compute it again for a metric given by $\Omega^2(x)g_{\mu\nu}$, where $\Omega(x)$ is an arbitrary nonvanishing function of spacetime, you get the same answer. For this reason it is often known as the "conformal tensor."

After this large amount of formalism, it might be time to step back and think about what curvature means for some simple examples. First notice that, according to (3.85), in 1, 2, 3 and 4 dimensions there are 0, 1, 6 and 20 components of the curvature tensor, respectively. (Everything we say about the curvature in these examples refers to the curvature associated with the Christoffel connection, and therefore the metric.) This means that one-dimensional manifolds (such as S^1) are never curved; the intuition you have that tells you that a circle is curved comes from thinking of it embedded in a certain flat two-dimensional plane. (There is something called "extrinsic curvature," which characterizes the way something is embedded in a higher dimensional space. Our notion of curvature is "intrinsic," and has nothing to do with such embeddings.)

The distinction between intrinsic and extrinsic curvature is also important in two dimensions, where the curvature has one independent component. (In fact, all of the information

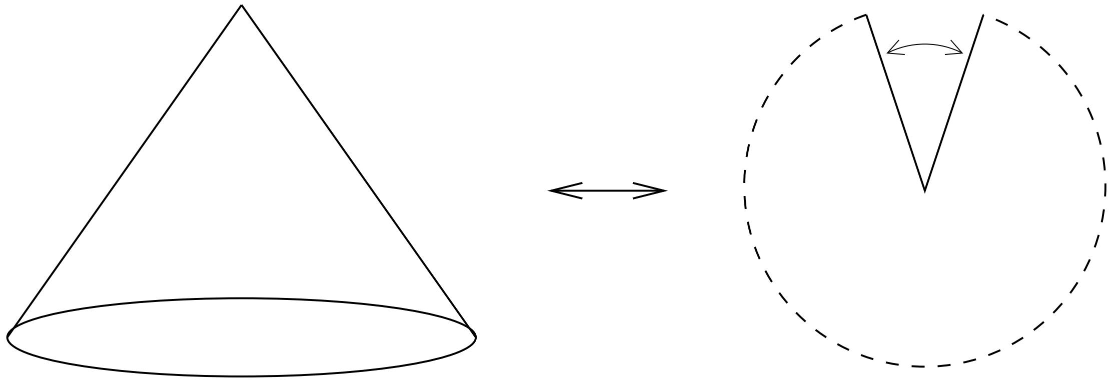


about the curvature is contained in the single component of the Ricci scalar.) Consider a cylinder, $\mathbf{R} \times S^1$. Although this looks curved from our point of view, it should be clear that we can put a metric on the cylinder whose components are constant in an appropriate coordinate system — simply unroll it and use the induced metric from the plane. In this metric, the cylinder is flat. (There is also nothing to stop us from introducing a different metric in which the cylinder is not flat, but the point we are trying to emphasize is that it can be made flat in some metric.) The same story holds for the torus:

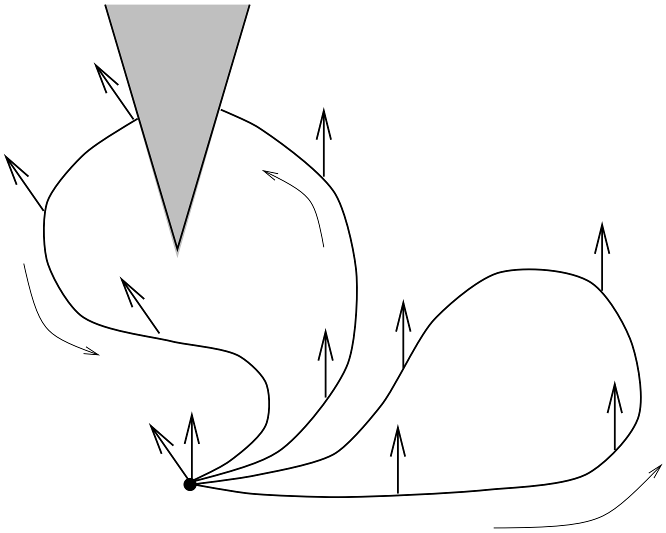


We can think of the torus as a square region of the plane with opposite sides identified (in other words, $S^1 \times S^1$), from which it is clear that it can have a flat metric even though it looks curved from the embedded point of view.

A cone is an example of a two-dimensional manifold with nonzero curvature at exactly one point. We can see this also by unrolling it; the cone is equivalent to the plane with a "deficit angle" removed and opposite sides identified:



In the metric inherited from this description as part of the flat plane, the cone is flat everywhere but at its vertex. This can be seen by considering parallel transport of a vector around various loops; if a loop does not enclose the vertex, there will be no overall transformation, whereas a loop that does enclose the vertex (say, just one time) will lead to a rotation by an angle which is just the deficit angle.



Our favorite example is of course the two-sphere, with metric

$$ds^2 = a^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\ , \tag{3.100}$$

where a is the radius of the sphere (thought of as embedded in $\mathbf{R}^3$). Without going through the details, the nonzero connection coefficients are

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta\ . \end{aligned} \tag{3.101}$$

Let's compute a promising component of the Riemann tensor:

$$R^\theta{}_{\phi\theta\phi} = \partial_\theta\Gamma^\theta_{\phi\phi} - \partial_\phi\Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi}$$

$$
\begin{aligned}
&= (\sin^2\theta - \cos^2\theta) - (0) + (0) - (-\sin\theta\cos\theta)(\cot\theta) \\
&= \sin^2\theta \ .
\end{aligned}
\tag{3.102}
$$

(The notation is obviously imperfect, since the Greek letter λ is a dummy index which is summed over, while the Greek letters θ and ϕ represent specific coordinates.) Lowering an index, we have

$$
\begin{aligned}
R_{\theta\phi\theta\phi} &= g_{\theta\lambda} R^{\lambda}{}_{\phi\theta\phi} \\
&= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\
&= a^2 \sin^2\theta \ .
\end{aligned}
\tag{3.103}
$$

It is easy to check that all of the components of the Riemann tensor either vanish or are related to this one by symmetry. We can go on to compute the Ricci tensor via $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$. We obtain

$$
\begin{aligned}
R_{\theta\theta} &= g^{\phi\phi} R_{\phi\theta\phi\theta} = 1 \\
R_{\theta\phi} &= R_{\phi\theta} = 0 \\
R_{\phi\phi} &= g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta \ .
\end{aligned}
\tag{3.104}
$$

The Ricci scalar is similarly straightforward:

$$
R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2} \ . \tag{3.105}
$$

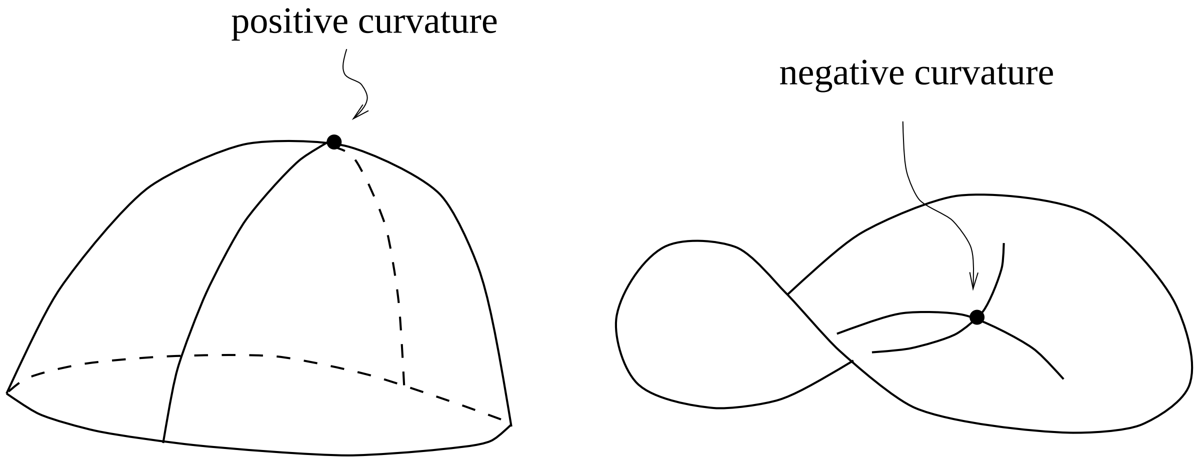
Therefore the Ricci scalar, which for a two-dimensional manifold completely characterizes the curvature, is a constant over this two-sphere. This is a reflection of the fact that the manifold is "maximally symmetric," a concept we will define more precisely later (although it means what you think it should). In any number of dimensions the curvature of a maximally symmetric space satisfies (for some constant a)

$$
R_{\rho\sigma\mu\nu} = a^{-2}(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}) \ , \tag{3.106}
$$

which you may check is satisfied by this example.

Notice that the Ricci scalar is not only constant for the two-sphere, it is manifestly positive. We say that the sphere is "positively curved" (of course a convention or two came into play, but fortunately our conventions conspired so that spaces which everyone agrees to call positively curved actually have a positive Ricci scalar). From the point of view of someone living on a manifold which is embedded in a higher-dimensional Euclidean space, if they are sitting at a point of positive curvature the space curves away from them in the same way in any direction, while in a negatively curved space it curves away in opposite directions. Negatively curved spaces are therefore saddle-like.

Enough fun with examples. There is one more topic we have to cover before introducing general relativity itself: geodesic deviation. You have undoubtedly heard that the defining



property of Euclidean (flat) geometry is the parallel postulate: initially parallel lines remain parallel forever. Of course in a curved space this is not true; on a sphere, certainly, initially parallel geodesics will eventually cross. We would like to quantify this behavior for an arbitrary curved space.

The problem is that the notion of "parallel" does not extend naturally from flat to curved spaces. Instead what we will do is to construct a one-parameter family of geodesics, $\gamma_s(t)$. That is, for each $s \in \mathbf{R}$, γ_s is a geodesic parameterized by the affine parameter t. The collection of these curves defines a smooth two-dimensional surface (embedded in a manifold M of arbitrary dimensionality). The coordinates on this surface may be chosen to be s and t, provided we have chosen a family of geodesics which do not cross. The entire surface is the set of points $x^\mu(s,t) \in M$. We have two natural vector fields: the tangent vectors to the geodesics,

$$T^\mu = \frac{\partial x^\mu}{\partial t} , \tag{3.107}$$

and the "deviation vectors"

$$S^\mu = \frac{\partial x^\mu}{\partial s} . \tag{3.108}$$

This name derives from the informal notion that S^μ points from one geodesic towards the neighboring ones.

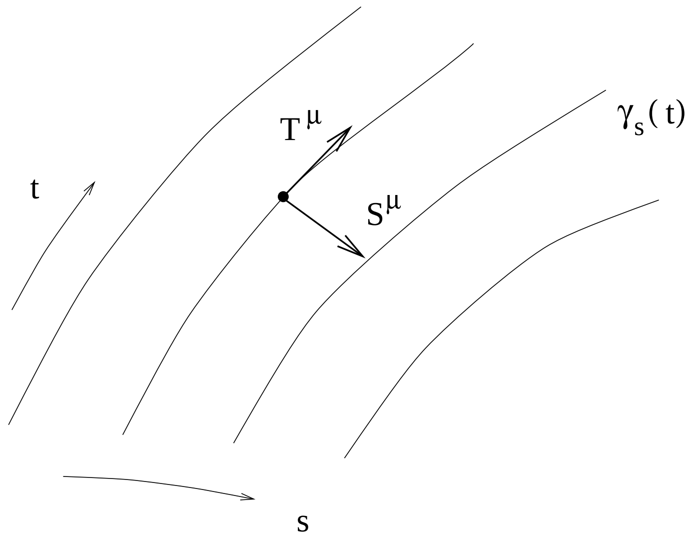
The idea that S^μ points from one geodesic to the next inspires us to define the "relative velocity of geodesics,"

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu , \tag{3.109}$$

and the "relative acceleration of geodesics,"

$$a^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu . \tag{3.110}$$

You should take the names with a grain of salt, but these vectors are certainly well-defined.



Since S and T are basis vectors adapted to a coordinate system, their commutator vanishes:

$$[S,T]=0\ .$$

We would like to consider the conventional case where the torsion vanishes, so from (3.70) we then have

$$S^\rho\nabla_\rho T^\mu = T^\rho\nabla_\rho S^\mu\ . \tag{3.111}$$

With this in mind, let's compute the acceleration:

$$\begin{aligned}
a^\mu &= T^\rho\nabla_\rho(T^\sigma\nabla_\sigma S^\mu)\\
&= T^\rho\nabla_\rho(S^\sigma\nabla_\sigma T^\mu)\\
&= (T^\rho\nabla_\rho S^\sigma)(\nabla_\sigma T^\mu)+T^\rho S^\sigma\nabla_\rho\nabla_\sigma T^\mu\\
&= (S^\rho\nabla_\rho T^\sigma)(\nabla_\sigma T^\mu)+T^\rho S^\sigma(\nabla_\sigma\nabla_\rho T^\mu+R^\mu{}_{\nu\rho\sigma}T^\nu)\\
&= (S^\rho\nabla_\rho T^\sigma)(\nabla_\sigma T^\mu)+S^\sigma\nabla_\sigma(T^\rho\nabla_\rho T^\mu)-(S^\sigma\nabla_\sigma T^\rho)\nabla_\rho T^\mu+R^\mu{}_{\nu\rho\sigma}T^\nu T^\rho S^\sigma\\
&= R^\mu{}_{\nu\rho\sigma}T^\nu T^\rho S^\sigma\ .
\end{aligned} \tag{3.112}$$

Let's think about this line by line. The first line is the definition of a^μ, and the second line comes directly from (3.111). The third line is simply the Leibniz rule. The fourth line replaces a double covariant derivative by the derivatives in the opposite order plus the Riemann tensor. In the fifth line we use Leibniz again (in the opposite order from usual), and then we cancel two identical terms and notice that the term involving $T^\rho\nabla_\rho T^\mu$ vanishes because T^μ is the tangent vector to a geodesic. The result,

$$a^\mu = \frac{D^2}{dt^2}S^\mu = R^\mu{}_{\nu\rho\sigma}T^\nu T^\rho S^\sigma\ , \tag{3.113}$$

is known as the **geodesic deviation equation**. It expresses something that we might have expected: the relative acceleration between two neighboring geodesics is proportional to the curvature.

Physically, of course, the acceleration of neighboring geodesics is interpreted as a manifestation of gravitational tidal forces. This reminds us that we are very close to doing physics by now.

There is one last piece of formalism which it would be nice to cover before we move on to gravitation proper. What we will do is to consider once again (although much more concisely) the formalism of connections and curvature, but this time we will use sets of basis vectors in the tangent space which are *not* derived from any coordinate system. It will turn out that this slight change in emphasis reveals a different point of view on the connection and curvature, one in which the relationship to gauge theories in particle physics is much more transparent. In fact the concepts to be introduced are very straightforward, but the subject is a notational nightmare, so it looks more difficult than it really is.

Up until now we have been taking advantage of the fact that a natural basis for the tangent space T_p at a point p is given by the partial derivatives with respect to the coordinates at that point, $\hat{e}_{(\mu)} = \partial_\mu$. Similarly, a basis for the cotangent space T_p^* is given by the gradients of the coordinate functions, $\hat{\theta}^{(\mu)} = \mathrm{d}x^\mu$. There is nothing to stop us, however, from setting up any bases we like. Let us therefore imagine that at each point in the manifold we introduce a set of basis vectors $\hat{e}_{(a)}$ (indexed by a Latin letter rather than Greek, to remind us that they are not related to any coordinate system). We will choose these basis vectors to be "orthonormal", in a sense which is appropriate to the signature of the manifold we are working on. That is, if the canonical form of the metric is written η_{ab}, we demand that the inner product of our basis vectors be

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} \ , \tag{3.114}$$

where $g(\ ,\ )$ is the usual metric tensor. Thus, in a Lorentzian spacetime η_{ab} represents the Minkowski metric, while in a space with positive-definite metric it would represent the Euclidean metric. The set of vectors comprising an orthonormal basis is sometimes known as a **tetrad** (from Greek *tetras*, "a group of four") or **vielbein** (from the German for "many legs"). In different numbers of dimensions it occasionally becomes a *vierbein* (four), *dreibein* (three), *zweibein* (two), and so on. (Just as we cannot in general find coordinate charts which cover the entire manifold, we will often not be able to find a single set of smooth basis vector fields which are defined everywhere. As usual, we can overcome this problem by working in different patches and making sure things are well-behaved on the overlaps.)

The point of having a basis is that any vector can be expressed as a linear combination of basis vectors. Specifically, we can express our old basis vectors $\hat{e}_{(\mu)} = \partial_\mu$ in terms of the new ones:

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)} \ . \tag{3.115}$$

The components e^a_μ form an $n \times n$ invertible matrix. (In accord with our usual practice of blurring the distinction between objects and their components, we will refer to the e^a_μ as the tetrad or vielbein, and often in the plural as "vielbeins.") We denote their inverse by switching indices to obtain e^μ_a, which satisfy

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \ , \qquad e^a_\mu e^\mu_b = \delta^a_b \ . \tag{3.116}$$

These serve as the components of the vectors $\hat{e}_{(a)}$ in the coordinate basis:

$$\hat{e}_{(a)} = e^\mu_a \hat{e}_{(\mu)} \ . \tag{3.117}$$

In terms of the inverse vielbeins, (3.114) becomes

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} \ , \tag{3.118}$$

or equivalently

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \ . \tag{3.119}$$

This last equation sometimes leads people to say that the vielbeins are the "square root" of the metric.

We can similarly set up an orthonormal basis of one-forms in T^*_p, which we denote $\hat{\theta}^{(a)}$. They may be chosen to be compatible with the basis vectors, in the sense that

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a_b \ . \tag{3.120}$$

It is an immediate consequence of this that the orthonormal one-forms are related to their coordinate-based cousins $\hat{\theta}^{(\mu)} = \mathrm{d}x^\mu$ by

$$\hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)} \tag{3.121}$$

and

$$\hat{\theta}^{(a)} = e^a_\mu \hat{\theta}^{(\mu)} \ . \tag{3.122}$$

The vielbeins e^a_μ thus serve double duty as the components of the coordinate basis vectors in terms of the orthonormal basis vectors, and as components of the orthonormal basis one-forms in terms of the coordinate basis one-forms; while the inverse vielbeins serve as the components of the orthonormal basis vectors in terms of the coordinate basis, and as components of the coordinate basis one-forms in terms of the orthonormal basis.

Any other vector can be expressed in terms of its components in the orthonormal basis. If a vector V is written in the coordinate basis as $V^\mu \hat{e}_{(\mu)}$ and in the orthonormal basis as $V^a \hat{e}_{(a)}$, the sets of components will be related by

$$V^a = e^a_\mu V^\mu \ . \tag{3.123}$$

So the vielbeins allow us to "switch from Latin to Greek indices and back." The nice property of tensors, that there is usually only one sensible thing to do based on index placement, is of great help here. We can go on to refer to multi-index tensors in either basis, or even in terms of mixed components:

$$V^a{}_b = e^a_\mu V^\mu{}_b = e^\nu_b V^a{}_\nu = e^a_\mu e^\nu_b V^\mu{}_\nu \ . \tag{3.124}$$

Looking back at (3.118), we see that the components of the metric tensor in the orthonormal basis are just those of the flat metric, η_{ab}. (For this reason the Greek indices are sometimes referred to as "curved" and the Latin ones as "flat.") In fact we can go so far as to raise and lower the Latin indices using the flat metric and its inverse η^{ab}. You can check for yourself that everything works okay (*e.g.*, that the lowering an index with the metric commutes with changing from orthonormal to coordinate bases).

By introducing a new set of basis vectors and one-forms, we necessitate a return to our favorite topic of transformation properties. We've been careful all along to emphasize that the tensor transformation law was only an indirect outcome of a coordinate transformation; the real issue was a change of basis. Now that we have non-coordinate bases, these bases can be changed independently of the coordinates. The only restriction is that the orthonormality property (3.114) be preserved. But we know what kind of transformations preserve the flat metric — in a Euclidean signature metric they are orthogonal transformations, while in a Lorentzian signature metric they are Lorentz transformations. We therefore consider changes of basis of the form

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda_{a'}{}^a(x)\hat{e}_{(a)} \ , \tag{3.125}$$

where the matrices $\Lambda_{a'}{}^a(x)$ represent position-dependent transformations which (at each point) leave the canonical form of the metric unaltered:

$$\Lambda_{a'}{}^a \Lambda_{b'}{}^b \eta_{ab} = \eta_{a'b'} \ . \tag{3.126}$$

In fact these matrices correspond to what in flat space we called the inverse Lorentz transformations (which operate on basis vectors); as before we also have ordinary Lorentz transformations $\Lambda^{a'}{}_a$, which transform the basis one-forms. As far as components are concerned, as before we transform upper indices with $\Lambda^{a'}{}_a$ and lower indices with $\Lambda_{a'}{}^a$.

So we now have the freedom to perform a Lorentz transformation (or an ordinary Euclidean rotation, depending on the signature) at every point in space. These transformations are therefore called **local Lorentz transformations**, or LLT's. We still have our usual freedom to make changes in coordinates, which are called **general coordinate transformations**, or GCT's. Both can happen at the same time, resulting in a mixed tensor transformation law:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda_{b'}{}^b \frac{\partial x^\nu}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu} \ . \tag{3.127}$$

Translating what we know about tensors into non-coordinate bases is for the most part merely a matter of sticking vielbeins in the right places. The crucial exception comes when we begin to differentiate things. In our ordinary formalism, the covariant derivative of a tensor is given by its partial derivative plus correction terms, one for each index, involving the tensor and the connection coefficients. The same procedure will continue to be true for the non-coordinate basis, but we replace the ordinary connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ by the **spin connection**, denoted $\omega_\mu{}^a{}_b$. Each Latin index gets a factor of the spin connection in the usual way:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c \ . \tag{3.128}$$

(The name "spin connection" comes from the fact that this can be used to take covariant derivatives of spinors, which is actually impossible using the conventional connection coefficients.) In the presence of mixed Latin and Greek indices we get terms of both kinds.

The usual demand that a tensor be independent of the way it is written allows us to derive a relationship between the spin connection, the vielbeins, and the $\Gamma^{\nu}_{\mu\lambda}$'s. Consider the covariant derivative of a vector X, first in a purely coordinate basis:

$$\begin{aligned}\nabla X &= (\nabla_\mu X^\nu)\mathrm{d}x^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma^{\nu}_{\mu\lambda} X^\lambda)\mathrm{d}x^\mu \otimes \partial_\nu \ .\end{aligned} \tag{3.129}$$

Now find the same object in a mixed basis, and convert into the coordinate basis:

$$\begin{aligned}\nabla X &= (\nabla_\mu X^a)\mathrm{d}x^\mu \otimes \hat{e}_{(a)} \\ &= (\partial_\mu X^a + \omega_\mu{}^a{}_b X^b)\mathrm{d}x^\mu \otimes \hat{e}_{(a)} \\ &= (\partial_\mu(e^a_\nu X^\nu) + \omega_\mu{}^a{}_b e^b_\lambda X^\lambda)\mathrm{d}x^\mu \otimes (e^\sigma_a \partial_\sigma) \\ &= e^\sigma_a (e^a_\nu \partial_\mu X^\nu + X^\nu \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\lambda X^\lambda)\mathrm{d}x^\mu \otimes \partial_\sigma \\ &= (\partial_\mu X^\nu + e^\nu_a \partial_\mu e^a_\lambda X^\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b X^\lambda)\mathrm{d}x^\mu \otimes \partial_\nu \ .\end{aligned} \tag{3.130}$$

Comparison with (3.129) reveals

$$\Gamma^{\nu}_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b \ , \tag{3.131}$$

or equivalently

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^{\nu}_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \ . \tag{3.132}$$

A bit of manipulation allows us to write this relation as the vanishing of the covariant derivative of the vielbein,

$$\nabla_\mu e^a_\nu = 0 \ , \tag{3.133}$$

which is sometimes known as the "tetrad postulate." Note that this is always true; we did not need to assume anything about the connection in order to derive it. Specifically, we did not need to assume that the connection was metric compatible or torsion free.

Since the connection may be thought of as something we need to fix up the transformation law of the covariant derivative, it should come as no surprise that the spin connection does not itself obey the tensor transformation law. Actually, under GCT's the one lower Greek index does transform in the right way, as a one-form. But under LLT's the spin connection transforms inhomogeneously, as

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda_{b'}{}^b \omega_\mu{}^a{}_b - \Lambda_{b'}{}^c \partial_\mu \Lambda^{a'}{}_c \ . \tag{3.134}$$

You are encouraged to check for yourself that this results in the proper transformation of the covariant derivative.

So far we have done nothing but empty formalism, translating things we already knew into a new notation. But the work we are doing does buy us two things. The first, which we already alluded to, is the ability to describe spinor fields on spacetime and take their covariant derivatives; we won't explore this further right now. The second is a change in viewpoint, in which we can think of various tensors as tensor-valued differential forms. For example, an object like $X_\mu{}^a$, which we think of as a $(1,1)$ tensor written with mixed indices, can also be thought of as a "vector-valued one-form." It has one lower Greek index, so we think of it as a one-form, but for each value of the lower index it is a vector. Similarly a tensor $A_{\mu\nu}{}^a{}_b$, antisymmetric in μ and ν, can be thought of as a "$(1,1)$-tensor-valued two-form." Thus, any tensor with some number of antisymmetric lower Greek indices and some number of Latin indices can be thought of as a differential form, but taking values in the tensor bundle. (Ordinary differential forms are simply scalar-valued forms.) The usefulness of this viewpoint comes when we consider exterior derivatives. If we want to think of $X_\mu{}^a$ as a vector-valued one-form, we are tempted to take its exterior derivative:

$$(\mathrm{d}X)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a \ . \tag{3.135}$$

It is easy to check that this object transforms like a two-form (that is, according to the transformation law for $(0,2)$ tensors) under GCT's, but not as a vector under LLT's (the Lorentz transformations depend on position, which introduces an inhomogeneous term into the transformation law). But we can fix this by judicious use of the spin connection, which can be thought of as a one-form. (Not a tensor-valued one-form, due to the nontensorial transformation law (3.134).) Thus, the object

$$(\mathrm{d}X)_{\mu\nu}{}^a + (\omega \wedge X)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \omega_\mu{}^a{}_b X_\nu{}^b - \omega_\nu{}^a{}_b X_\mu{}^b \ , \tag{3.136}$$

as you can verify at home, transforms as a proper tensor.

An immediate application of this formalism is to the expressions for the torsion and curvature, the two tensors which characterize any given connection. The torsion, with two antisymmetric lower indices, can be thought of as a vector-valued two-form $T_{\mu\nu}{}^a$. The curvature, which is always antisymmetric in its last two indices, is a $(1,1)$-tensor-valued two-form, $R^a{}_{b\mu\nu}$. Using our freedom to suppress indices on differential forms, we can write the defining relations for these two tensors as

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{3.137}$$

and

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \tag{3.138}$$

These are known as the **Maurer-Cartan structure equations**. They are equivalent to the usual definitions; let's go through the exercise of showing this for the torsion, and you can check the curvature for yourself. We have

$$\begin{aligned} T_{\mu\nu}{}^\lambda &= e^\lambda_a T_{\mu\nu}{}^a \\ &= e^\lambda_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b) \\ &= \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \ , \end{aligned} \tag{3.139}$$

which is just the original definition we gave. Here we have used (3.131), the expression for the $\Gamma^\lambda_{\mu\nu}$'s in terms of the vielbeins and spin connection. We can also express identities obeyed by these tensors as

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \tag{3.140}$$

and

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \ . \tag{3.141}$$

The first of these is the generalization of $R^\rho{}_{[\sigma\mu\nu]} = 0$, while the second is the Bianchi identity $\nabla_{[\lambda|}R^\rho{}_{\sigma|\mu\nu]} = 0$. (Sometimes both equations are called Bianchi identities.)

The form of these expressions leads to an almost irresistible temptation to define a "covariant-exterior derivative", which acts on a tensor-valued form by taking the ordinary exterior derivative and then adding appropriate terms with the spin connection, one for each Latin index. Although we won't do that here, it is okay to give in to this temptation, and in fact the right hand side of (3.137) and the left hand sides of (3.140) and (3.141) can be thought of as just such covariant-exterior derivatives. But be careful, since (3.138) cannot; you can't take any sort of covariant derivative of the spin connection, since it's not a tensor.

So far our equations have been true for general connections; let's see what we get for the Christoffel connection. The torsion-free requirement is just that (3.137) vanish; this does not lead immediately to any simple statement about the coefficients of the spin connection. Metric compatibility is expressed as the vanishing of the covariant derivative of the metric: $\nabla g = 0$. We can see what this leads to when we express the metric in the orthonormal basis, where its components are simply η_{ab}:

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac}$$

$$= \quad -\omega_{\mu ab} - \omega_{\mu ba} \ . \tag{3.142}$$

Then setting this equal to zero implies

$$\omega_{\mu ab} = -\omega_{\mu ba} \ . \tag{3.143}$$

Thus, metric compatibility is equivalent to the antisymmetry of the spin connection in its Latin indices. (As before, such a statement is only sensible if both indices are either upstairs or downstairs.) These two conditions together allow us to express the spin connection in terms of the vielbeins. There is an explicit formula which expresses this solution, but in practice it is easier to simply solve the torsion-free condition

$$\omega^{ab} \wedge e_b = -\mathrm{d}e^a \ , \tag{3.144}$$

using the asymmetry of the spin connection, to find the individual components.

We now have the means to compare the formalism of connections and curvature in Riemannian geometry to that of gauge theories in particle physics. (This is an aside, which is hopefully comprehensible to everybody, but not an essential ingredient of the course.) In both situations, the fields of interest live in vector spaces which are assigned to each point in spacetime. In Riemannian geometry the vector spaces include the tangent space, the cotangent space, and the higher tensor spaces constructed from these. In gauge theories, on the other hand, we are concerned with "internal" vector spaces. The distinction is that the tangent space and its relatives are intimately associated with the manifold itself, and were naturally defined once the manifold was set up; an internal vector space can be of any dimension we like, and has to be defined as an independent addition to the manifold. In math lingo, the union of the base manifold with the internal vector spaces (defined at each point) is a **fiber bundle**, and each copy of the vector space is called the "fiber" (in perfect accord with our definition of the tangent bundle).

Besides the base manifold (for us, spacetime) and the fibers, the other important ingredient in the definition of a fiber bundle is the "structure group," a Lie group which acts on the fibers to describe how they are sewn together on overlapping coordinate patches. Without going into details, the structure group for the tangent bundle in a four-dimensional spacetime is generally $\mathrm{GL}(4, \mathbf{R})$, the group of real invertible 4×4 matrices; if we have a Lorentzian metric, this may be reduced to the Lorentz group $\mathrm{SO}(3,1)$. Now imagine that we introduce an internal three-dimensional vector space, and sew the fibers together with ordinary rotations; the structure group of this new bundle is then $\mathrm{SO}(3)$. A field that lives in this bundle might be denoted $\phi^A(x^\mu)$, where A runs from one to three; it is a three-vector (an internal one, unrelated to spacetime) for each point on the manifold. We have freedom to choose the basis in the fibers in any way we wish; this means that "physical quantities" should be left invariant under local $\mathrm{SO}(3)$ transformations such as

$$\phi^A(x^\mu) \to \phi^{A'}(x^\mu) = O^{A'}{}_A(x^\mu)\phi^A(x^\mu) \ , \tag{3.145}$$

where $O^{A'}{}_A(x^\mu)$ is a matrix in SO(3) which depends on spacetime. Such transformations are known as **gauge transformations**, and theories invariant under them are called "gauge theories."

For the most part it is not hard to arrange things such that physical quantities are invariant under gauge transformations. The one difficulty arises when we consider partial derivatives, $\partial_\mu \phi^A$. Because the matrix $O^{A'}{}_A(x^\mu)$ depends on spacetime, it will contribute an unwanted term to the transformation of the partial derivative. By now you should be able to guess the solution: introduce a connection to correct for the inhomogeneous term in the transformation law. We therefore define a connection on the fiber bundle to be an object $A_\mu{}^A{}_B$, with two "group indices" and one spacetime index. Under GCT's it transforms as a one-form, while under gauge transformations it transforms as

$$A_\mu{}^{A'}{}_{B'} = O^{A'}{}_A O_{B'}{}^B A_\mu{}^A{}_B - O_{B'}{}^C \partial_\mu O^{A'}{}_C \ . \tag{3.146}$$

(Beware: our conventions are so drastically different from those in the particle physics literature that I won't even try to get them straight.) With this transformation law, the "gauge covariant derivative"

$$D_\mu \phi^A = \partial_\mu \phi^A + A_\mu{}^A{}_B \phi^B \tag{3.147}$$

transforms "tensorially" under gauge transformations, as you are welcome to check. (In ordinary electromagnetism the connection is just the conventional vector potential. No indices are necessary, because the structure group U(1) is one-dimensional.)

It is clear that this notion of a connection on an internal fiber bundle is very closely related to the connection on the tangent bundle, especially in the orthonormal-frame picture we have been discussing. The transformation law (3.146), for example, is exactly the same as the transformation law (3.134) for the spin connection. We can also define a curvature or "field strength" tensor which is a two-form,

$$F^A{}_B = \mathrm{d}A^A{}_B + A^A{}_C \wedge A^C{}_B \ , \tag{3.148}$$

in exact correspondence with (3.138). We can parallel transport things along paths, and there is a construction analogous to the parallel propagator; the trace of the matrix obtained by parallel transporting a vector around a closed curve is called a "Wilson loop."

We could go on in the development of the relationship between the tangent bundle and internal vector bundles, but time is short and we have other fish to fry. Let us instead finish by emphasizing the important *difference* between the two constructions. The difference stems from the fact that the tangent bundle is closely related to the base manifold, while other fiber bundles are tacked on after the fact. It makes sense to say that a vector in the tangent space at p "points along a path" through p; but this makes no sense for an internal vector bundle. There is therefore no analogue of the coordinate basis for an internal space —

partial derivatives along curves have nothing to do with internal vectors. It follows in turn that there is nothing like the vielbeins, which relate orthonormal bases to coordinate bases. The torsion tensor, in particular, is only defined for a connection on the tangent bundle, not for any gauge theory connections; it can be thought of as the covariant exterior derivative of the vielbein, and no such construction is available on an internal bundle. You should appreciate the relationship between the different uses of the notion of a connection, without getting carried away.

4 Gravitation

Having paid our mathematical dues, we are now prepared to examine the physics of gravitation as described by general relativity. This subject falls naturally into two pieces: how the curvature of spacetime acts on matter to manifest itself as "gravity", and how energy and momentum influence spacetime to create curvature. In either case it would be legitimate to start at the top, by stating outright the laws governing physics in curved spacetime and working out their consequences. Instead, we will try to be a little more motivational, starting with basic physical principles and attempting to argue that these lead naturally to an almost unique physical theory.

The most basic of these physical principles is the Principle of Equivalence, which comes in a variety of forms. The earliest form dates from Galileo and Newton, and is known as the **Weak Equivalence Principle**, or WEP. The WEP states that the "inertial mass" and "gravitational mass" of any object are equal. To see what this means, think about Newton's Second Law. This relates the force exerted on an object to the acceleration it undergoes, setting them proportional to each other with the constant of proportionality being the inertial mass m_i:

$$\mathbf{f} = m_i\mathbf{a} \ . \tag{4.1}$$

The inertial mass clearly has a universal character, related to the resistance you feel when you try to push on the object; it is the same constant no matter what kind of force is being exerted. We also have the law of gravitation, which states that the gravitational force exerted on an object is proportional to the gradient of a scalar field Φ, known as the gravitational potential. The constant of proportionality in this case is called the gravitational mass m_g:

$$\mathbf{f}_g = -m_g\nabla\Phi \ . \tag{4.2}$$

On the face of it, m_g has a very different character than m_i; it is a quantity specific to the gravitational force. If you like, it is the "gravitational charge" of the body. Nevertheless, Galileo long ago showed (apocryphally by dropping weights off of the Leaning Tower of Pisa, actually by rolling balls down inclined planes) that the response of matter to gravitation was universal — every object falls at the same rate in a gravitational field, independent of the composition of the object. In Newtonian mechanics this translates into the WEP, which is simply

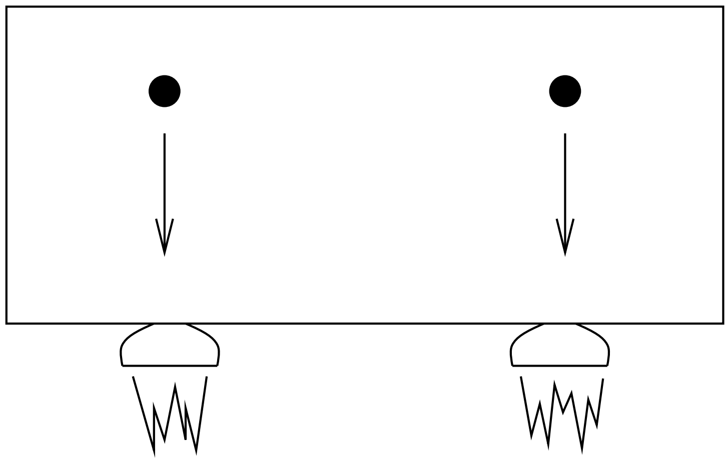
$$m_i = m_g \tag{4.3}$$

for any object. An immediate consequence is that the behavior of freely-falling test particles is universal, independent of their mass (or any other qualities they may have); in fact we have

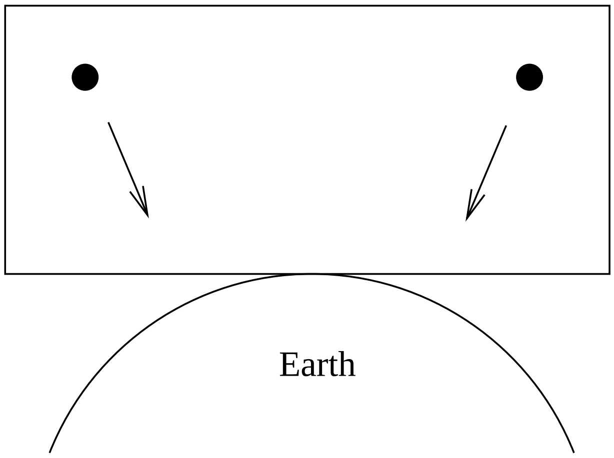
$$\mathbf{a} = -\nabla\Phi \ . \tag{4.4}$$

The universality of gravitation, as implied by the WEP, can be stated in another, more popular, form. Imagine that we consider a physicist in a tightly sealed box, unable to observe the outside world, who is doing experiments involving the motion of test particles, for example to measure the local gravitational field. Of course she would obtain different answers if the box were sitting on the moon or on Jupiter than she would on the Earth. But the answers would also be different if the box were accelerating at a constant velocity; this would change the acceleration of the freely-falling particles with respect to the box. The WEP implies that there is no way to disentangle the effects of a gravitational field from those of being in a uniformly accelerating frame, simply by observing the behavior of freely-falling particles. This follows from the universality of gravitation; it would be possible to distinguish between uniform acceleration and an electromagnetic field, by observing the behavior of particles with different charges. But with gravity it is impossible, since the "charge" is necessarily proportional to the (inertial) mass.

To be careful, we should limit our claims about the impossibility of distinguishing gravity from uniform acceleration by restricting our attention to "small enough regions of spacetime." If the sealed box were sufficiently big, the gravitational field would change from place to place in an observable way, while the effect of acceleration is always in the same direction. In a rocket ship or elevator, the particles always fall straight down:



In a very big box in a gravitational field, however, the particles will move toward the center of the Earth (for example), which might be a different direction in different regions:



The WEP can therefore be stated as "the laws of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime." In larger regions of spacetime there will be inhomogeneities in the gravitational field, which will lead to tidal forces which can be detected.

After the advent of special relativity, the concept of mass lost some of its uniqueness, as it became clear that mass was simply a manifestation of energy and momentum ($E = mc^2$ and all that). It was therefore natural for Einstein to think about generalizing the WEP to something more inclusive. His idea was simply that there should be no way whatsoever for the physicist in the box to distinguish between uniform acceleration and an external gravitational field, no matter what experiments she did (not only by dropping test particles). This reasonable extrapolation became what is now known as the **Einstein Equivalence Principle**, or EEP: "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field."

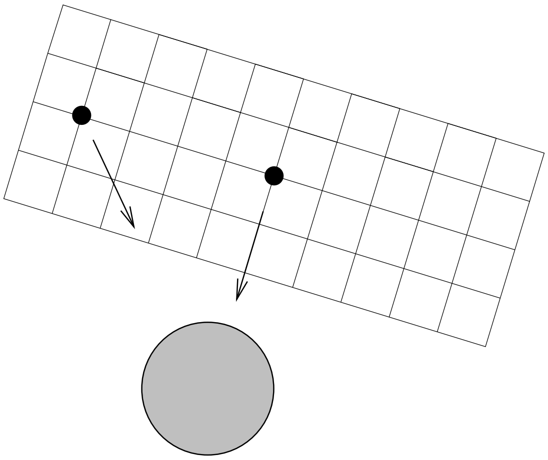
In fact, it is hard to imagine theories which respect the WEP but violate the EEP. Consider a hydrogen atom, a bound state of a proton and an electron. Its mass is actually less than the sum of the masses of the proton and electron considered individually, because there is a negative binding energy — you have to put energy into the atom to separate the proton and electron. According to the WEP, the gravitational mass of the hydrogen atom is therefore less than the sum of the masses of its constituents; the gravitational field couples to electromagnetism (which holds the atom together) in exactly the right way to make the gravitational mass come out right. This means that not only must gravity couple to rest mass universally, but to all forms of energy and momentum — which is practically the claim of the EEP. It is possible to come up with counterexamples, however; for example, we could imagine a theory of gravity in which freely falling particles began to rotate as they moved through a gravitational field. Then they could fall along the same paths as they would in an accelerated frame (thereby satisfying the WEP), but you could nevertheless detect the existence of the gravitational field (in violation of the EEP). Such theories seem contrived, but there is no law of nature which forbids them.

Sometimes a distinction is drawn between "gravitational laws of physics" and "non-gravitational laws of physics," and the EEP is defined to apply only to the latter. Then one defines the "Strong Equivalence Principle" (SEP) to include all of the laws of physics, gravitational and otherwise. I don't find this a particularly useful distinction, and won't belabor it. For our purposes, the EEP (or simply "the principle of equivalence") includes all of the laws of physics.

It is the EEP which implies (or at least suggests) that we should attribute the action of gravity to the curvature of spacetime. Remember that in special relativity a prominent role is played by inertial frames — while it was not possible to single out some frame of reference as uniquely "at rest", it was possible to single out a family of frames which were "unaccelerated" (inertial). The acceleration of a charged particle in an electromagnetic field was therefore uniquely defined with respect to these frames. The EEP, on the other hand, implies that gravity is inescapable — there is no such thing as a "gravitationally neutral object" with respect to which we can measure the acceleration due to gravity. It follows that "the acceleration due to gravity" is not something which can be reliably defined, and therefore is of little use.

Instead, it makes more sense to *define* "unaccelerated" as "freely falling," and that is what we shall do. This point of view is the origin of the idea that gravity is not a "force" — a force is something which leads to acceleration, and our definition of zero acceleration is "moving freely in the presence of whatever gravitational field happens to be around."

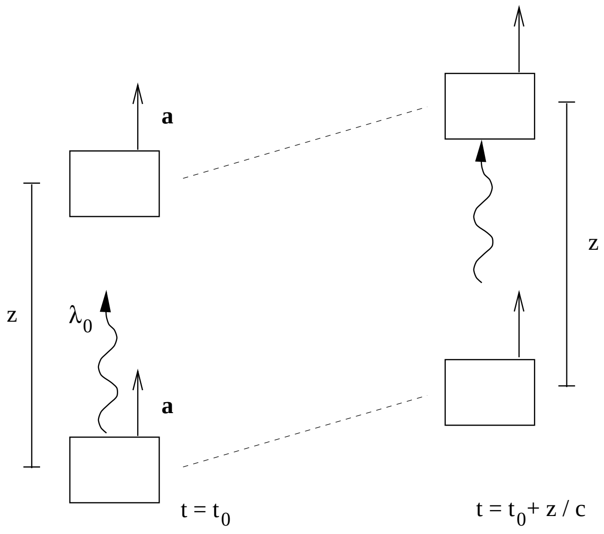
This seemingly innocuous step has profound implications for the nature of spacetime. In SR, we had a procedure for starting at some point and constructing an inertial frame which stretched throughout spacetime, by joining together rigid rods and attaching clocks to them. But, again due to inhomogeneities in the gravitational field, this is no longer possible. If we start in some freely-falling state and build a large structure out of rigid rods, at some distance away freely-falling objects will look like they are "accelerating" with respect to this reference frame, as shown in the figure on the next page.



The solution is to retain the notion of inertial frames, but to discard the hope that they can be uniquely extended throughout space and time. Instead we can define **locally inertial frames**, those which follow the motion of freely falling particles in small enough regions of spacetime. (Every time we say "small enough regions", purists should imagine a limiting procedure in which we take the appropriate spacetime volume to zero.) This is the best we can do, but it forces us to give up a good deal. For example, we can no longer speak with confidence about the relative velocity of far away objects, since the inertial reference frames appropriate to those objects are independent of those appropriate to us.

So far we have been talking strictly about physics, without jumping to the conclusion that spacetime should be described as a curved manifold. It should be clear, however, why such a conclusion is appropriate. The idea that the laws of special relativity should be obeyed in sufficiently small regions of spacetime, and further that local inertial frames can be established in such regions, corresponds to our ability to construct Riemann normal coordinates at any one point on a manifold — coordinates in which the metric takes its canonical form and the Christoffel symbols vanish. The impossibility of comparing velocities (vectors) at widely separated regions corresponds to the path-dependence of parallel transport on a curved manifold. These considerations were enough to give Einstein the idea that gravity was a manifestation of spacetime curvature. But in fact we can be even more persuasive. (It is impossible to "prove" that gravity should be thought of as spacetime curvature, since scientific hypotheses can only be falsified, never verified [and not even really falsified, as Thomas Kuhn has famously argued]. But there is nothing to be dissatisfied with about convincing plausibility arguments, if they lead to empirically successful theories.)

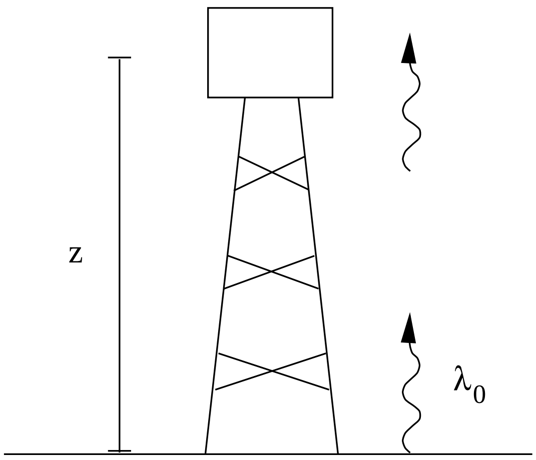
Let's consider one of the celebrated predictions of the EEP, the gravitational redshift. Consider two boxes, a distance z apart, moving (far away from any matter, so we assume in the absence of any gravitational field) with some constant acceleration a. At time t_0 the trailing box emits a photon of wavelength λ_0.



The boxes remain a constant distance apart, so the photon reaches the leading box after a time $\Delta t = z/c$ in the reference frame of the boxes. In this time the boxes will have picked up an additional velocity $\Delta v = a\Delta t = az/c$. Therefore, the photon reaching the lead box will be redshifted by the conventional Doppler effect by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2} . \tag{4.5}$$

(We assume $\Delta v/c$ is small, so we only work to first order.) According to the EEP, the same thing should happen in a uniform gravitational field. So we imagine a tower of height z sitting on the surface of a planet, with a_g the strength of the gravitational field (what Newton would have called the "acceleration due to gravity").



This situation is supposed to be indistinguishable from the previous one, from the point of view of an observer in a box at the top of the tower (able to detect the emitted photon, but otherwise unable to look outside the box). Therefore, a photon emitted from the ground with wavelength λ_0 should be redshifted by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2} \ . \tag{4.6}$$

This is the famous gravitational redshift. Notice that it is a direct consequence of the EEP, not of the details of general relativity. It has been verified experimentally, first by Pound and Rebka in 1960. They used the Mössbauer effect to measure the change in frequency in γ-rays as they traveled from the ground to the top of Jefferson Labs at Harvard.

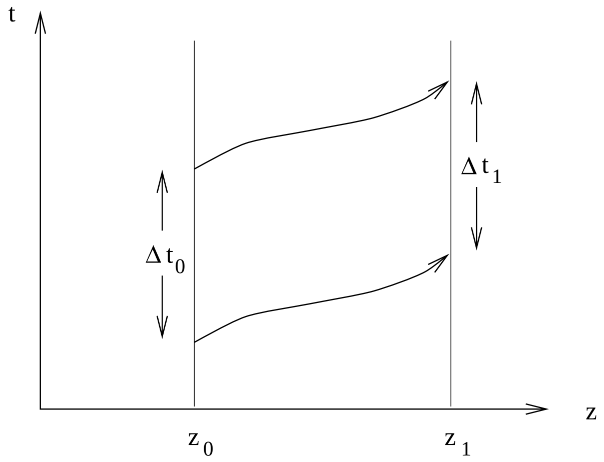
The formula for the redshift is more often stated in terms of the Newtonian potential Φ, where $\mathbf{a}_g = \nabla\Phi$. (The sign is changed with respect to the usual convention, since we are thinking of $\mathbf{a}_g$ as the acceleration of the reference frame, not of a particle with respect to this reference frame.) A non-constant gradient of Φ is like a time-varying acceleration, and the equivalent net velocity is given by integrating over the time between emission and absorption of the photon. We then have

$$\begin{aligned} \frac{\Delta\lambda}{\lambda_0} &= \frac{1}{c}\int \nabla\Phi \ dt \\ &= \frac{1}{c^2}\int \partial_z\Phi \ dz \\ &= \Delta\Phi \ , \end{aligned} \tag{4.7}$$

where $\Delta\Phi$ is the total change in the gravitational potential, and we have once again set $c = 1$. This simple formula for the gravitational redshift continues to be true in more general circumstances. Of course, by using the Newtonian potential at all, we are restricting our domain of validity to weak gravitational fields, but that is usually completely justified for observable effects.

The gravitational redshift leads to another argument that we should consider spacetime as curved. Consider the same experimental setup that we had before, now portrayed on the spacetime diagram on the next page.

The physicist on the ground emits a beam of light with wavelength λ_0 from a height z_0, which travels to the top of the tower at height z_1. The time between when the beginning of any single wavelength of the light is emitted and the end of that same wavelength is emitted is $\Delta t_0 = \lambda_0/c$, and the same time interval for the absorption is $\Delta t_1 = \lambda_1/c$. Since we imagine that the gravitational field is not varying with time, the paths through spacetime followed by the leading and trailing edge of the single wave must be precisely congruent. (They are represented by some generic curved paths, since we do not pretend that we know just what the paths will be.) Simple geometry tells us that the times Δt_0 and Δt_1 must be the same. But of course they are not; the gravitational redshift implies that $\Delta t_1 > \Delta t_0$. (Which we can interpret as "the clock on the tower appears to run more quickly.") The fault lies with



"simple geometry"; a better description of what happens is to imagine that spacetime is curved.

All of this should constitute more than enough motivation for our claim that, in the presence of gravity, spacetime should be thought of as a curved manifold. Let us now take this to be true and begin to set up how physics works in a curved spacetime. The principle of equivalence tells us that the laws of physics, in small enough regions of spacetime, look like those of special relativity. We interpret this in the language of manifolds as the statement that these laws, when written in Riemannian normal coordinates x^μ based at some point p, are described by equations which take the same form as they would in flat space. The simplest example is that of freely-falling (unaccelerated) particles. In flat space such particles move in straight lines; in equations, this is expressed as the vanishing of the second derivative of the parameterized path $x^\mu(\lambda)$:

$$\frac{d^2x^\mu}{d\lambda^2} = 0\ . \tag{4.8}$$

According to the EEP, exactly this equation should hold in curved space, as long as the coordinates x^μ are RNC's. What about some other coordinate system? As it stands, (4.8) is not an equation between tensors. However, there is a unique tensorial equation which reduces to (4.8) when the Christoffel symbols vanish; it is

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0\ . \tag{4.9}$$

Of course, this is simply the geodesic equation. In general relativity, therefore, free particles move along geodesics; we have mentioned this before, but now you know why it is true.

As far as free particles go, we have argued that curvature of spacetime is necessary to describe gravity; we have not yet shown that it is sufficient. To do so, we can show how the usual results of Newtonian gravity fit into the picture. We define the "Newtonian limit" by three requirements: the particles are moving slowly (with respect to the speed of light), the gravitational field is weak (can be considered a perturbation of flat space), and the field is also static (unchanging with time). Let us see what these assumptions do to the geodesic equation, taking the proper time τ as an affine parameter. "Moving slowly" means that

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau} \ , \tag{4.10}$$

so the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \ . \tag{4.11}$$

Since the field is static, the relevant Christoffel symbols Γ^μ_{00} simplify:

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \ .\end{aligned} \tag{4.12}$$

Finally, the weakness of the gravitational field allows us to decompose the metric into the Minkowski form plus a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| << 1 \ . \tag{4.13}$$

(We are working in Cartesian coordinates, so $\eta_{\mu\nu}$ is the canonical form of the metric. The "smallness condition" on the metric perturbation $h_{\mu\nu}$ doesn't really make sense in other coordinates.) From the definition of the inverse metric, $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$, we find that to first order in h,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{4.14}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. In fact, we can use the Minkowski metric to raise and lower indices on an object of any definite order in h, since the corrections would only contribute at higher orders.

Putting it all together, we find

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \ . \tag{4.15}$$

The geodesic equation (4.11) is therefore

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2 \ . \tag{4.16}$$

Using $\partial_0 h_{00} = 0$, the $\mu = 0$ component of this is just

$$\frac{d^2t}{d\tau^2} = 0 \ . \tag{4.17}$$

That is, $\frac{dt}{d\tau}$ is constant. To examine the spacelike components of (4.16), recall that the spacelike components of $\eta^{\mu\nu}$ are just those of a 3×3 identity matrix. We therefore have

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00} \ . \tag{4.18}$$

Dividing both sides by $\left(\frac{dt}{d\tau}\right)^2$ has the effect of converting the derivative on the left-hand side from τ to t, leaving us with

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \ . \tag{4.19}$$

This begins to look a great deal like Newton's theory of gravitation. In fact, if we compare this equation to (4.4), we find that they are the same once we identify

$$h_{00} = -2\Phi \ , \tag{4.20}$$

or in other words

$$g_{00} = -(1 + 2\Phi) \ . \tag{4.21}$$

Therefore, we have shown that the curvature of spacetime is indeed sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form (4.21). It remains, of course, to find field equations for the metric which imply that this is the form taken, and that for a single gravitating body we recover the Newtonian formula

$$\Phi = -\frac{GM}{r} \ , \tag{4.22}$$

but that will come soon enough.

Our next task is to show how the remaining laws of physics, beyond those governing freely-falling particles, adapt to the curvature of spacetime. The procedure essentially follows the paradigm established in arguing that free particles move along geodesics. Take a law of physics in flat space, traditionally written in terms of partial derivatives and the flat metric. According to the equivalence principle this law will hold in the presence of gravity, as long as we are in Riemannian normal coordinates. Translate the law into a relationship between tensors; for example, change partial derivatives to covariant ones. In RNC's this version of the law will reduce to the flat-space one, but tensors are coordinate-independent objects, so the tensorial version must hold in any coordinate system.

This procedure is sometimes given a name, the **Principle of Covariance**. I'm not sure that it deserves its own name, since it's really a consequence of the EEP plus the requirement that the laws of physics be independent of coordinates. (The requirement that laws of physics be independent of coordinates is essentially impossible to even imagine being untrue. Given some experiment, if one person uses one coordinate system to predict a result and another one uses a different coordinate system, they had better agree.) Another name is the "comma-goes-to-semicolon rule", since at a typographical level the thing you have to do is replace partial derivatives (commas) with covariant ones (semicolons).

We have already implicitly used the principle of covariance (or whatever you want to call it) in deriving the statement that free particles move along geodesics. For the most part, it is very simple to apply it to interesting cases. Consider for example the formula for conservation of energy in flat spacetime, $\partial_\mu T^{\mu\nu} = 0$. The adaptation to curved spacetime is immediate:

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{4.23}$$

This equation expresses the conservation of energy in the presence of a gravitational field.

Unfortunately, life is not always so easy. Consider Maxwell's equations in special relativity, where it would seem that the principle of covariance can be applied in a straightforward way. The inhomogeneous equation $\partial_\mu F^{\nu\mu} = 4\pi J^\nu$ becomes

$$\nabla_\mu F^{\nu\mu} = 4\pi J^\nu \ , \tag{4.24}$$

and the homogeneous one $\partial_{[\mu} F_{\nu\lambda]} = 0$ becomes

$$\nabla_{[\mu} F_{\nu\lambda]} = 0 \ . \tag{4.25}$$

On the other hand, we could also write Maxwell's equations in flat space in terms of differential forms as

$$\mathrm{d}(*F) = 4\pi(*J) \ , \tag{4.26}$$

and

$$\mathrm{d}F = 0 \ . \tag{4.27}$$

These are already in perfectly tensorial form, since we have shown that the exterior derivative is a well-defined tensor operator regardless of what the connection is. We therefore begin to worry a little bit; what is the guarantee that the process of writing a law of physics in tensorial form gives a unique answer? In fact, as we have mentioned earlier, the differential forms versions of Maxwell's equations should be taken as fundamental. Nevertheless, in this case it happens to make no difference, since in the absence of torsion (4.26) is identical to (4.24), and (4.27) is identical to (4.25); the symmetric part of the connection doesn't contribute. Similarly, the definition of the field strength tensor in terms of the potential A_μ can be written either as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ , \tag{4.28}$$

or equally well as

$$F = \mathrm{d}A \ . \tag{4.29}$$

The worry about uniqueness is a real one, however. Imagine that two vector fields X^μ and Y^ν obey a law in flat space given by

$$Y^\mu \partial_\mu \partial_\nu X^\nu = 0 \ . \tag{4.30}$$

The problem in writing this as a tensor equation should be clear: the partial derivatives can be commuted, but covariant derivatives cannot. If we simply replace the partials in (4.30) by covariant derivatives, we get a different answer than we would if we had first exchanged the order of the derivatives (leaving the equation in flat space invariant) and then replaced them. The difference is given by

$$Y^\mu \nabla_\mu \nabla_\nu X^\nu - Y^\mu \nabla_\nu \nabla_\mu X^\nu = -R_{\mu\nu} Y^\mu X^\nu \ . \tag{4.31}$$

The prescription for generalizing laws from flat to curved spacetimes does not guide us in choosing the order of the derivatives, and therefore is ambiguous about whether a term such as that in (4.31) should appear in the presence of gravity. (The problem of ordering covariant derivatives is similar to the problem of operator-ordering ambiguities in quantum mechanics.)

In the literature you can find various prescriptions for dealing with ambiguities such as this, most of which are sensible pieces of advice such as remembering to preserve gauge invariance for electromagnetism. But deep down the real answer is that there is no way to resolve these problems by pure thought alone; the fact is that there may be more than one way to adapt a law of physics to curved space, and ultimately only experiment can decide between the alternatives.

In fact, let us be honest about the principle of equivalence: it serves as a useful guideline, but it does not deserve to be treated as a fundamental principle of nature. From the modern point of view, we do not expect the EEP to be rigorously true. Consider the following alternative version of (4.24):

$$\nabla_\mu[(1+\alpha R)F^{\nu\mu}] = 4\pi J^\nu \ , \tag{4.32}$$

where R is the Ricci scalar and α is some coupling constant. If this equation correctly described electrodynamics in curved spacetime, it would be possible to measure R even in an arbitrarily small region, by doing experiments with charged particles. The equivalence principle therefore demands that $\alpha = 0$. But otherwise this is a perfectly respectable equation, consistent with charge conservation and other desirable features of electromagnetism, which reduces to the usual equation in flat space. Indeed, in a world governed by quantum mechanics we expect all possible couplings between different fields (such as gravity and electromagnetism) that are consistent with the symmetries of the theory (in this case, gauge invariance). So why is it reasonable to set $\alpha = 0$? The real reason is one of scales. Notice that the Ricci tensor involves second derivatives of the metric, which is dimensionless, so R has dimensions of (length)$^{-2}$ (with $c = 1$). Therefore α must have dimensions of (length)^2. But since the coupling represented by α is of gravitational origin, the only reasonable expectation for the relevant length scale is

$$\alpha \sim l_P^2 \ , \tag{4.33}$$

where l_P is the Planck length

$$l_P = \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6 \times 10^{-33} \text{ cm} , \tag{4.34}$$

where $\hbar$ is of course Planck's constant. So the length scale corresponding to this coupling is extremely small, and for any conceivable experiment we expect the typical scale of variation for the gravitational field to be much larger. Therefore the reason why this equivalence-principle-violating term can be safely ignored is simply because αR is probably a fantastically small number, far out of the reach of any experiment. On the other hand, we might as well keep an open mind, since our expectations are not always borne out by observation.

Having established how physical laws govern the behavior of fields and objects in a curved spacetime, we can complete the establishment of general relativity proper by introducing Einstein's field equations, which govern how the metric responds to energy and momentum. We will actually do this in two ways: first by an informal argument close to what Einstein himself was thinking, and then by starting with an action and deriving the corresponding equations of motion.

The informal argument begins with the realization that we would like to find an equation which supersedes the Poisson equation for the Newtonian potential:

$$\nabla^2\Phi = 4\pi G\rho , \tag{4.35}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ is the mass density. (The explicit form of Φ given in (4.22) is one solution of (4.35), for the case of a pointlike mass distribution.) What characteristics should our sought-after equation possess? On the left-hand side of (4.35) we have a second-order differential operator acting on the gravitational potential, and on the right-hand side a measure of the mass distribution. A relativistic generalization should take the form of an equation between tensors. We know what the tensor generalization of the mass density is; it's the energy-momentum tensor $T_{\mu\nu}$. The gravitational potential, meanwhile, should get replaced by the metric tensor. We might therefore guess that our new equation will have $T_{\mu\nu}$ set proportional to some tensor which is second-order in derivatives of the metric. In fact, using (4.21) for the metric in the Newtonian limit and $T_{00} = \rho$, we see that in this limit we are looking for an equation that predicts

$$\nabla^2 h_{00} = -8\pi G T_{00} , \tag{4.36}$$

but of course we want it to be completely tensorial.

The left-hand side of (4.36) does not obviously generalize to a tensor. The first choice might be to act the D'Alembertian $\Box = \nabla^\mu\nabla_\mu$ on the metric $g_{\mu\nu}$, but this is automatically zero by metric compatibility. Fortunately, there is an obvious quantity which is not zero and is constructed from second derivatives (and first derivatives) of the metric: the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$. It doesn't have the right number of indices, but we can contract it to form the Ricci tensor $R_{\mu\nu}$, which does (and is symmetric to boot). It is therefore reasonable to guess that the gravitational field equations are

$$R_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{4.37}$$

for some constant κ. In fact, Einstein did suggest this equation at one point. There is a problem, unfortunately, with conservation of energy. According to the Principle of Equivalence, the statement of energy-momentum conservation in curved spacetime should be

$$\nabla^\mu T_{\mu\nu} = 0 \ , \tag{4.38}$$

which would then imply

$$\nabla^\mu R_{\mu\nu} = 0 \ . \tag{4.39}$$

This is certainly not true in an arbitrary geometry; we have seen from the Bianchi identity (3.94) that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \ . \tag{4.40}$$

But our proposed field equation implies that $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T$, so taking these together we have

$$\nabla_\mu T = 0 \ . \tag{4.41}$$

The covariant derivative of a scalar is just the partial derivative, so (4.41) is telling us that T is constant throughout spacetime. This is highly implausible, since $T = 0$ in vacuum while $T > 0$ in matter. We have to try harder.

(Actually we are cheating slightly, in taking the equation $\nabla^\mu T_{\mu\nu} = 0$ so seriously. If as we said, the equivalence principle is only an approximate guide, we could imagine that there are nonzero terms on the right-hand side involving the curvature tensor. Later we will be more precise and argue that they are strictly zero.)

Of course we don't have to try much harder, since we already know of a symmetric $(0, 2)$ tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \ , \tag{4.42}$$

which always obeys $\nabla^\mu G_{\mu\nu} = 0$. We are therefore led to propose

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.43}$$

as a field equation for the metric. This equation satisfies all of the obvious requirements; the right-hand side is a covariant expression of the energy and momentum density in the form of a symmetric and conserved $(0,2)$ tensor, while the left-hand side is a symmetric and conserved $(0,2)$ tensor constructed from the metric and its first and second derivatives. It only remains to see whether it actually reproduces gravity as we know it.

To answer this, note that contracting both sides of (4.43) yields (in four dimensions)

$$R = -\kappa T \ , \tag{4.44}$$

and using this we can rewrite (4.43) as

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \ . \tag{4.45}$$

This is the same equation, just written slightly differently. We would like to see if it predicts Newtonian gravity in the weak-field, time-independent, slowly-moving-particles limit. In this limit the rest energy $\rho = T_{00}$ will be much larger than the other terms in $T_{\mu\nu}$, so we want to focus on the $\mu = 0$, $\nu = 0$ component of (4.45). In the weak-field limit, we write (in accordance with (4.13) and (4.14))

$$\begin{aligned} g_{00} &= -1 + h_{00} \ , \\ g^{00} &= -1 - h_{00} \ . \end{aligned} \tag{4.46}$$

The trace of the energy-momentum tensor, to lowest nontrivial order, is

$$T = g^{00}T_{00} = -T_{00} \ . \tag{4.47}$$

Plugging this into (4.45), we get

$$R_{00} = \frac{1}{2}\kappa T_{00} \ . \tag{4.48}$$

This is an equation relating derivatives of the metric to the energy density. To find the explicit expression in terms of the metric, we need to evaluate $R_{00} = R^\lambda{}_{0\lambda 0}$. In fact we only need $R^i{}_{0i0}$, since $R^0{}_{000} = 0$. We have

$$R^i{}_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0} \ . \tag{4.49}$$

The second term here is a time derivative, which vanishes for static fields. The third and fourth terms are of the form $(\Gamma)^2$, and since Γ is first-order in the metric perturbation these contribute only at second order, and can be neglected. We are left with $R^i{}_{0j0} = \partial_j\Gamma^i_{00}$. From this we get

$$\begin{aligned} R_{00} &= R^i{}_{0i0} \\ &= \partial_i\left(\frac{1}{2}g^{i\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00})\right) \\ &= -\frac{1}{2}\eta^{ij}\partial_i\partial_j h_{00} \end{aligned}$$

$$= -\frac{1}{2}\nabla^2 h_{00} \ . \tag{4.50}$$

Comparing to (4.48), we see that the 00 component of (4.43) in the Newtonian limit predicts

$$\nabla^2 h_{00} = -\kappa T_{00} \ . \tag{4.51}$$

But this is exactly (4.36), if we set $\kappa = 8\pi G$.

So our guess seems to have worked out. With the normalization fixed by comparison with the Newtonian limit, we can present **Einstein's equations** for general relativity:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{4.52}$$

These tell us how the curvature of spacetime reacts to the presence of energy-momentum. Einstein, you may have heard, thought that the left-hand side was nice and geometrical, while the right-hand side was somewhat less compelling.

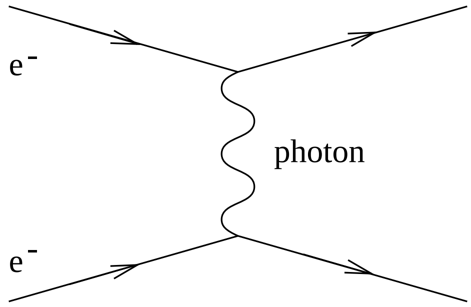
Einstein's equations may be thought of as second-order differential equations for the metric tensor field $g_{\mu\nu}$. There are ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ represents four constraints on the functions $R_{\mu\nu}$, so there are only six truly independent equations in (4.52). In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system x^μ it should also be a solution in any other coordinate system $x^{\mu'}$. This means that there are four unphysical degrees of freedom in $g_{\mu\nu}$ (represented by the four functions $x^{\mu'}(x^\mu)$), and we should expect that Einstein's equations only constrain the six coordinate-independent degrees of freedom.

As differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric as well. The equations are also nonlinear, so that two known solutions cannot be superposed to find a third. It is therefore very difficult to solve Einstein's equations in any sort of generality, and it is usually necessary to make some simplifying assumptions. Even in vacuum, where we set the energy-momentum tensor to zero, the resulting equations (from (4.45))

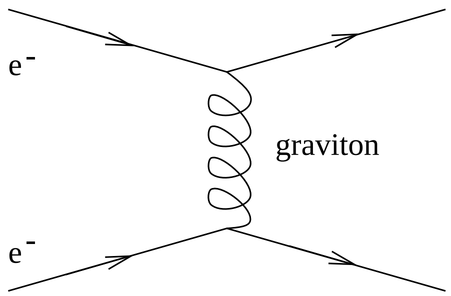
$$R_{\mu\nu} = 0 \tag{4.53}$$

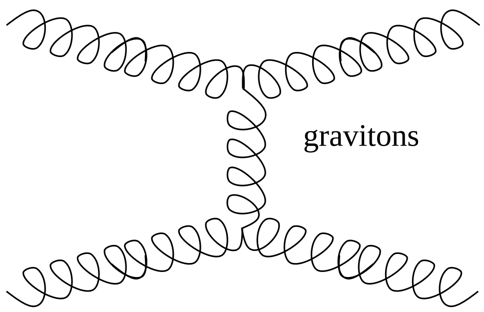
can be very difficult to solve. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will talk later on about how symmetries of the metric make life easier.

The nonlinearity of general relativity is worth remarking on. In Newtonian gravity the potential due to two point masses is simply the sum of the potentials for each mass, but clearly this does not carry over to general relativity (outside the weak-field limit). There is a physical reason for this, namely that in GR the gravitational field couples to itself. This can be thought of as a consequence of the equivalence principle — if gravitation did not couple to itself, a "gravitational atom" (two particles bound by their mutual gravitational attraction) would have a different inertial mass (due to the negative binding energy) than gravitational mass. From a particle physics point of view this can be expressed in terms of Feynman diagrams. The electromagnetic interaction between two electrons can be thought of as due to exchange of a virtual photon:



But there is no diagram in which two photons exchange another photon between themselves; electromagnetism is linear. The gravitational interaction, meanwhile, can be thought of as due to exchange of a virtual graviton (a quantized perturbation of the metric). The nonlinearity manifests itself as the fact that both electrons and gravitons (and anything else) can exchange virtual gravitons, and therefore exert a gravitational force:





There is nothing profound about this feature of gravity; it is shared by most gauge theories, such as quantum chromodynamics, the theory of the strong interactions. (Electromagnetism is actually the exception; the linearity can be traced to the fact that the relevant gauge group, U(1), is abelian.) But it does represent a departure from the Newtonian theory. (Of course this quantum mechanical language of Feynman diagrams is somewhat inappropriate for GR, which has not [yet] been successfully quantized, but the diagrams are just a convenient shorthand for remembering what interactions exist in the theory.)

To increase your confidence that Einstein's equations as we have derived them are indeed the correct field equations for the metric, let's see how they can be derived from a more modern viewpoint, starting from an action principle. (In fact the equations were first derived by Hilbert, not Einstein, and Hilbert did it using the action principle. But he had been inspired by Einstein's previous papers on the subject, and Einstein himself derived the equations independently, so they are rightly named after Einstein. The action, however, is rightly called the Hilbert action.) The action should be the integral over spacetime of a Lagrange density ("Lagrangian" for short, although strictly speaking the Lagrangian is the integral over space of the Lagrange density):

$$S_H = \int d^n x \mathcal{L}_H \ . \tag{4.54}$$

The Lagrange density is a tensor density, which can be written as $\sqrt{-g}$ times a scalar. What scalars can we make out of the metric? Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar R. What we did not show, but is nevertheless true, is that any nontrivial tensor made from the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the only independent scalar constructed from the metric, which is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$\mathcal{L}_H = \sqrt{-g} R \ . \tag{4.55}$$

The equations of motion should come from varying the action with respect to the metric. In fact let us consider variations with respect to the inverse metric $g^{\mu\nu}$, which are slightly easier but give an equivalent set of equations. Using $R = g^{\mu\nu} R_{\mu\nu}$, in general we will have

$$\begin{aligned} \delta S &= \int d^n x \left[\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} \right] \\ &= (\delta S)_1 + (\delta S)_2 + (\delta S)_3 \ . \end{aligned} \tag{4.56}$$

The second term $(\delta S)_2$ is already in the form of some expression times $\delta g^{\mu\nu}$; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma} \Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu) \ . \tag{4.57}$$

The variation of this with respect the metric can be found first varying the connection with respect to the metric, and then substituting into this expression. Let us however consider arbitrary variations of the connection, by replacing

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} \ . \tag{4.58}$$

The variation $\delta\Gamma^{\rho}_{\nu\mu}$ is the difference of two connections, and therefore is itself a tensor. We can thus take its covariant derivative,

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} \ . \tag{4.59}$$

Given this expression (and a small amount of labor) it is easy to show that

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \ . \tag{4.60}$$

You can check this yourself. Therefore, the contribution of the first term in (4.56) to δS can be written

$$\begin{aligned} (\delta S)_1 &= \int d^n x\sqrt{-g}\ g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \\ &= \int d^n x\sqrt{-g}\ \nabla_\sigma\left[g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu})\right] \ , \end{aligned} \tag{4.61}$$

where we have used metric compatibility and relabeled some dummy indices. But now we have the integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity which we can set to zero by making the variation vanish at infinity. (We haven't actually shown that Stokes's theorem, as mentioned earlier in terms of differential forms, can be thought of this way, but you can easily convince yourself it's true.) Therefore this term contributes nothing to the total variation.

To make sense of the $(\delta S)_3$ term we need to use the following fact, true for any matrix M:

$$\mathrm{Tr}(\ln M) = \ln(\det M) \ . \tag{4.62}$$

Here, $\ln M$ is defined by $\exp(\ln M) = M$. (For numbers this is obvious, for matrices it's a little less straightforward.) The variation of this identity yields

$$\mathrm{Tr}(M^{-1}\delta M) = \frac{1}{\det M}\delta(\det M) \ . \tag{4.63}$$

Here we have used the cyclic property of the trace to allow us to ignore the fact that M^{-1} and δM may not commute. Now we would like to apply this to the inverse metric, $M = g^{\mu\nu}$. Then $\det M = g^{-1}$ (where $g = \det g_{\mu\nu}$), and

$$\delta(g^{-1}) = \frac{1}{g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{4.64}$$

Now we can just plug in:

$$\begin{aligned}
\delta\sqrt{-g} &= \delta[(-g^{-1})^{-1/2}] \\
&= -\frac{1}{2}(-g^{-1})^{-3/2}\delta(-g^{-1}) \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \ .
\end{aligned} \tag{4.65}$$

Hearkening back to (4.56), and remembering that $(\delta S)_1$ does not contribute, we find

$$\delta S = \int d^n x\sqrt{-g}\ \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \ . \tag{4.66}$$

This should vanish for arbitrary variations, so we are led to Einstein's equations in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \ . \tag{4.67}$$

The fact that this simple action leads to the same vacuum field equations as we had previously arrived at by more informal arguments certainly reassures us that we are doing something right. What we would really like, however, is to get the non-vacuum field equations as well. That means we consider an action of the form

$$S = \frac{1}{8\pi G}S_H + S_M \ , \tag{4.68}$$

where S_M is the action for matter, and we have presciently normalized the gravitational action (although the proper normalization is somewhat convention-dependent). Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \ , \tag{4.69}$$

and we recover Einstein's equations if we can set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \ . \tag{4.70}$$

What makes us think that we can make such an identification? In fact (4.70) turns out to be the best way to define a symmetric energy-momentum tensor. The tricky part is to show that it is conserved, which is in fact automatically true, but which we will not justify until the next section.

We say that (4.70) provides the "best" definition of the energy-momentum tensor because it is not the only one you will find. In flat Minkowski space, there is an alternative definition which is sometimes given in books on electromagnetism or field theory. In this context energy-momentum conservation arises as a consequence of symmetry of the Lagrangian under spacetime translations. *Noether's theorem* states that every symmetry of a Lagrangian implies the existence of a conservation law; invariance under the four spacetime translations leads to a tensor $S^{\mu\nu}$ which obeys $\partial_\mu S^{\mu\nu} = 0$ (four relations, one for each value of ν). The details can be found in Wald or in any number of field theory books. Applying Noether's procedure to a Lagrangian which depends on some fields ψ^i and their first derivatives $\partial_\mu \psi^i$, we obtain

$$S^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi^i)} \partial^\nu \psi^i - \eta^{\mu\nu} \mathcal{L} \ , \tag{4.71}$$

where a sum over i is implied. You can check that this tensor is conserved by virtue of the equations of motion of the matter fields. $S^{\mu\nu}$ often goes by the name "canonical energy-momentum tensor"; however, there are a number of reasons why it is more convenient for us to use (4.70). First and foremost, (4.70) is in fact what appears on the right hand side of Einstein's equations when they are derived from an action, and it is not always possible to generalize (4.71) to curved spacetime. But even in flat space (4.70) has its advantages; it is manifestly symmetric, and also guaranteed to be gauge invariant, neither of which is true for (4.71). We will therefore stick with (4.70) as the definition of the energy-momentum tensor.

Sometimes it is useful to think about Einstein's equations without specifying the theory of matter from which $T_{\mu\nu}$ is derived. This leaves us with a great deal of arbitrariness; consider for example the question "What metrics obey Einstein's equations?" In the absence of some constraints on $T_{\mu\nu}$, the answer is "any metric at all"; simply take the metric of your choice, compute the Einstein tensor $G_{\mu\nu}$ for this metric, and then demand that $T_{\mu\nu}$ be equal to $G_{\mu\nu}$. (It will automatically be conserved, by the Bianchi identity.) Our real concern is with the existence of solutions to Einstein's equations in the presence of "realistic" sources of energy and momentum, whatever that means. The most common property that is demanded of $T_{\mu\nu}$ is that it represent positive energy densities — no negative masses are allowed. In a locally inertial frame this requirement can be stated as $\rho = T_{00} \geq 0$. To turn this into a coordinate-independent statement, we ask that

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \ , \qquad \text{for all timelike vectors } V^\mu \ . \tag{4.72}$$

This is known as the **Weak Energy Condition**, or WEC. It seems like a fairly reasonable requirement, and many of the important theorems about solutions to general relativity (such as the singularity theorems of Hawking and Penrose) rely on this condition or something very close to it. Unfortunately it is not set in stone; indeed, it is straightforward to invent otherwise respectable classical field theories which violate the WEC, and almost impossible to invent a quantum field theory which obeys it. Nevertheless, it is legitimate to assume that the WEC holds in all but the most extreme conditions. (There are also stronger energy conditions, but they are even less true than the WEC, and we won't dwell on them.)

We have now justified Einstein's equations in two different ways: as the natural covariant generalization of Poisson's equation for the Newtonian gravitational potential, and as the result of varying the simplest possible action we could invent for the metric. The rest of the course will be an exploration of the consequences of these equations, but before we start on that road let us briefly explore ways in which the equations could be modified. There are an uncountable number of such ways, but we will consider four different possibilities: the introduction of a cosmological constant, higher-order terms in the action, gravitational scalar fields, and a nonvanishing torsion tensor.

The first possibility is the cosmological constant; George Gamow has quoted Einstein as calling this the biggest mistake of his life. Recall that in our search for the simplest possible action for gravity we noted that any nontrivial scalar had to be of at least second order in derivatives of the metric; at lower order all we can create is a constant. Although a constant does not by itself lead to very interesting dynamics, it has an important effect if we add it to the conventional Hilbert action. We therefore consider an action given by

$$S = \int d^n x \sqrt{-g}(R - 2\Lambda) \ , \tag{4.73}$$

where Λ is some constant. The resulting field equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{4.74}$$

and of course there would be an energy-momentum tensor on the right hand side if we had included an action for matter. Λ is the cosmological constant; it was originally introduced by Einstein after it became clear that there were no solutions to his equations representing a static cosmology (a universe unchanging with time on large scales) with a nonzero matter content. If the cosmological constant is tuned just right, it is possible to find a static solution, but it is unstable to small perturbations. Furthermore, once Hubble demonstrated that the universe is expanding, it became less important to find static solutions, and Einstein rejected his suggestion. Like Rasputin, however, the cosmological constant has proven difficult to kill off. If we like we can move the additional term in (4.74) to the right hand side, and think of it as a kind of energy-momentum tensor, with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ (it is automatically conserved by metric compatibility). Then Λ can be interpreted as the "energy density of the vacuum," a source of energy and momentum that is present even in the absence of matter fields. This interpretation is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In ordinary quantum mechanics, an harmonic oscillator with frequency ω and minimum classical energy $E_0 = 0$ upon quantization has a ground state with energy $E_0 = \frac{1}{2}\hbar\omega$. A quantized field can be thought of as a collection of an infinite number of harmonic oscillators, and each mode contributes to the ground state energy. The result is of course infinite, and must be appropriately regularized, for example by introducing a cutoff at high frequencies. The final vacuum energy, which is the regularized sum of the energies of the ground state oscillations of all the fields of the theory, has no good reason to be zero and in fact would be expected to have a natural scale

$$\Lambda \sim m_P^4 \ , \tag{4.75}$$

where the Planck mass m_P is approximately 10^{19} GeV, or 10^{-5} grams. Observations of the universe on large scales allow us to constrain the actual value of Λ, which turns out to be smaller than (4.75) by at least a factor of 10^{120}. This is the largest known discrepancy between theoretical estimate and observational constraint in physics, and convinces many people that the "cosmological constant problem" is one of the most important unsolved problems today. On the other hand the observations do not tell us that Λ is strictly zero, and in fact allow values that can have important consequences for the evolution of the universe. This mistake of Einstein's therefore continues to bedevil both physicists, who would like to understand why it is so small, and astronomers, who would like to determine whether it is really small enough to be ignored.

A somewhat less intriguing generalization of the Hilbert action would be to include scalars of more than second order in derivatives of the metric. We could imagine an action of the form

$$S = \int d^n x \sqrt{-g}(R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 g^{\mu\nu}\nabla_\mu R \nabla_\nu R + \cdots) \ , \tag{4.76}$$

where the α's are coupling constants and the dots represent every other scalar we can make from the curvature tensor, its contractions, and its derivatives. Traditionally, such terms have been neglected on the reasonable grounds that they merely complicate a theory which is already both aesthetically pleasing and empirically successful. However, there are at least three more substantive reasons for this neglect. First, as we shall see below, Einstein's equations lead to a well-posed initial value problem for the metric, in which "coordinates" and "momenta" specified at an initial time can be used to predict future evolution. With higher-derivative terms, we would require not only those data, but also some number of derivatives of the momenta. Second, the main source of dissatisfaction with general relativity on the part of particle physicists is that it cannot be renormalized (as far as we know), and Lagrangians with higher derivatives tend generally to make theories less renormalizable rather than more. Third, by the same arguments we used above when speaking about the limitations of the principle of equivalence, the extra terms in (4.76) should be suppressed (by powers of the Planck mass to some power) relative to the usual Hilbert term, and therefore would not be expected to be of any practical importance to the low-energy world. None of these reasons are completely persuasive, and indeed people continue to consider such theories, but for the most part these models do not attract a great deal of attention.

A set of models which does attract attention are known as **scalar-tensor theories** of gravity, since they involve both the metric tensor $g_{\mu\nu}$ and a fundamental scalar field, λ. The action can be written

$$S = \int d^n x \sqrt{-g} \left[f(\lambda) R + \frac{1}{2} g^{\mu\nu} (\partial_\mu \lambda)(\partial_\nu \lambda) - V(\lambda) \right] , \tag{4.77}$$

where $f(\lambda)$ and $V(\lambda)$ are functions which define the theory. Recall from (4.68) that the coefficient of the Ricci scalar in conventional GR is proportional to the inverse of Newton's constant G. In scalar-tensor theories, then, where this coefficient is replaced by some function of a field which can vary throughout spacetime, the "strength" of gravity (as measured by the local value of Newton's constant) will be different from place to place and time to time. In fact the most famous scalar-tensor theory, invented by Brans and Dicke and now named after them, was inspired by a suggestion of Dirac's that the gravitational constant varies with time. Dirac had noticed that there were some interesting numerical coincidences one could discover by taking combinations of cosmological numbers such as the Hubble constant H_0 (a measure of the expansion rate of the universe) and typical particle-physics parameters such as the mass of the pion, m_π. For example,

$$\frac{m_\pi^3}{H_0} \sim \frac{cG}{\hbar^2} . \tag{4.78}$$

If we assume for the moment that this relation is not simply an accident, we are faced with the problem that the Hubble "constant" actually changes with time (in most cosmological models), while the other quantities conventionally do not. Dirac therefore proposed that in fact G varied with time, in such a way as to maintain (4.78); satisfying this proposal was the motivation of Brans and Dicke. These days, experimental test of general relativity are sufficiently precise that we can state with confidence that, if Brans-Dicke theory is correct, the predicted change in G over space and time must be very small, much slower than that necessary to satisfy Dirac's hypothesis. (See Weinberg for details on Brans-Dicke theory and experimental tests.) Nevertheless there is still a great deal of work being done on other kinds of scalar-tensor theories, which turn out to be vital in superstring theory and may have important consequences in the very early universe.

As a final alternative to general relativity, we should mention the possibility that the connection really is not derived from the metric, but in fact has an independent existence as a fundamental field. We will leave it as an exercise for you to show that it is possible to consider the conventional action for general relativity but treat it as a function of both the metric $g_{\mu\nu}$ and a torsion-free connection $\Gamma^\lambda_{\rho\sigma}$, and the equations of motion derived from varying such an action with respect to the connection imply that $\Gamma^\lambda_{\rho\sigma}$ is actually the Christoffel connection associated with $g_{\mu\nu}$. We could drop the demand that the connection be torsion-free, in which case the torsion tensor could lead to additional propagating degrees of freedom. Without going into details, the basic reason why such theories do not receive much attention is simply because the torsion is itself a tensor; there is nothing to distinguish it from other, "non-gravitational" tensor fields. Thus, we do not really lose any generality by considering theories of torsion-free connections (which lead to GR) plus any number of tensor fields, which we can name what we like.

With the possibility in mind that one of these alternatives (or, more likely, something we have not yet thought of) is actually realized in nature, for the rest of the course we will work under the assumption that general relativity as based on Einstein's equations or the Hilbert action is the correct theory, and work out its consequences. These consequences, of course, are constituted by the solutions to Einstein's equations for various sources of energy and momentum, and the behavior of test particles in these solutions. Before considering specific solutions in detail, lets look more abstractly at the initial-value problem in general relativity.

In classical Newtonian mechanics, the behavior of a single particle is of course governed by $\mathbf{f} = m\mathbf{a}$. If the particle is moving under the influence of some potential energy field $\Phi(x)$, then the force is $\mathbf{f} = -\nabla\Phi$, and the particle obeys

$$m\frac{d^2x^i}{dt^2} = -\partial_i\Phi \ . \tag{4.79}$$

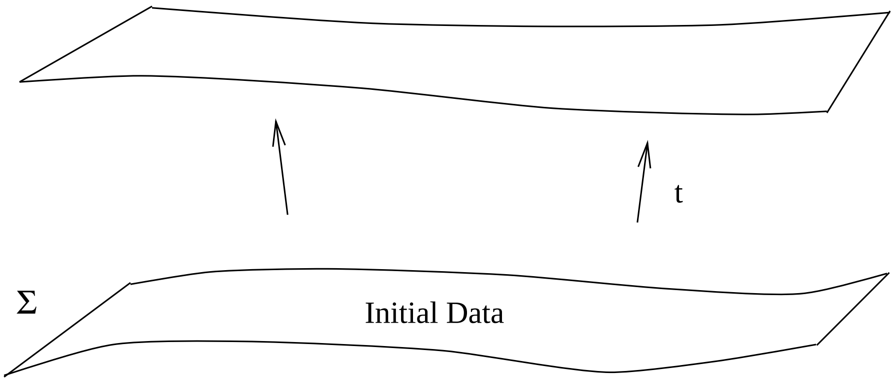
This is a second-order differential equation for $x^i(t)$, which we can recast as a system of two coupled first-order equations by introducing the momentum $\mathbf{p}$:

$$\begin{aligned} \frac{dp^i}{dt} &= -\partial_i\Phi \\ \frac{dx^i}{dt} &= \frac{1}{m}p^i \ . \end{aligned} \tag{4.80}$$

The initial-value problem is simply the procedure of specifying a "state" (x^i, p^i) which serves as a boundary condition with which (4.80) can be uniquely solved. You may think of (4.80) as allowing you, once you are given the coordinates and momenta at some time t, to evolve them forward an infinitesimal amount to a time $t + \delta t$, and iterate this procedure to obtain the entire solution.

We would like to formulate the analogous problem in general relativity. Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ are of course covariant; they don't single out a preferred notion of "time" through which a state can evolve. Nevertheless, we can by hand pick a spacelike hypersurface (or "slice") Σ, specify initial data on that hypersurface, and see if we can evolve uniquely from it to a hypersurface in the future. ("Hyper" because a constant-time slice in four dimensions will be three-dimensional, whereas "surfaces" are conventionally two-dimensional.) This process does violence to the manifest covariance of the theory, but if we are careful we should wind up with a formulation that is equivalent to solving Einstein's equations all at once throughout spacetime.

Since the metric is the fundamental variable, our first guess is that we should consider the values $g_{\mu\nu}|_\Sigma$ of the metric on our hypersurface to be the "coordinates" and the time



derivatives $\partial_t g_{\mu\nu}|_\Sigma$ (with respect to some specified time coordinate) to be the "momenta", which together specify the state. (There will also be coordinates and momenta for the matter fields, which we will not consider explicitly.) In fact the equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ do involve second derivatives of the metric with respect to time (since the connection involves first derivatives of the metric and the Einstein tensor involves first derivatives of the connection), so we seem to be on the right track. However, the Bianchi identity tells us that $\nabla_\mu G^{\mu\nu} = 0$. We can rewrite this equation as

$$\partial_0 G^{0\nu} = -\partial_i G^{i\nu} - \Gamma^\mu_{\mu\lambda} G^{\lambda\nu} - \Gamma^\nu_{\mu\lambda} G^{\mu\lambda} \ . \tag{4.81}$$

A close look at the right hand side reveals that there are no *third*-order time derivatives; therefore there cannot be any on the left hand side. Thus, although $G^{\mu\nu}$ as a whole involves second-order time derivatives of the metric, the specific components $G^{0\nu}$ do not. Of the ten independent components in Einstein's equations, the four represented by

$$G^{0\nu} = 8\pi G T^{0\nu} \tag{4.82}$$

cannot be used to evolve the initial data $(g_{\mu\nu}, \partial_t g_{\mu\nu})_\Sigma$. Rather, they serve as **constraints** on this initial data; we are not free to specify any combination of the metric and its time derivatives on the hypersurface Σ, since they must obey the relations (4.82). The remaining equations,

$$G^{ij} = 8\pi G T^{ij} \tag{4.83}$$

are the dynamical evolution equations for the metric. Of course, these are only six equations for the ten unknown functions $g_{\mu\nu}(x^\sigma)$, so the solution will inevitably involve a fourfold ambiguity. This is simply the freedom that we have already mentioned, to choose the four coordinate functions throughout spacetime.

It is a straightforward but unenlightening exercise to sift through (4.83) to find that not all second time derivatives of the metric appear. In fact we find that $\partial_t^2 g_{ij}$ appears in (4.83), but not $\partial_t^2 g_{0\nu}$. Therefore a "state" in general relativity will consist of a specification of the spacelike components of the metric $g_{ij}|_\Sigma$ and their first time derivatives $\partial_t g_{ij}|_\Sigma$ on the hypersurface Σ, from which we can determine the future evolution using (4.83), up to an unavoidable ambiguity in fixing the remaining components $g_{0\nu}$. The situation is precisely analogous to that in electromagnetism, where we know that no amount of initial data can suffice to determine the evolution uniquely since there will always be the freedom to perform a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$. In general relativity, then, coordinate transformations play a role reminiscent of gauge transformations in electromagnetism, in that they introduce ambiguity into the time evolution.

One way to cope with this problem is to simply "choose a gauge." In electromagnetism this means to place a condition on the vector potential A_μ, which will restrict our freedom to perform gauge transformations. For example we can choose Lorentz gauge, in which $\nabla_\mu A^\mu = 0$, or temporal gauge, in which $A_0 = 0$. We can do a similar thing in general relativity, by fixing our coordinate system. A popular choice is **harmonic gauge** (also known as Lorentz gauge and a host of other names), in which

$$\Box x^\mu = 0 \ . \tag{4.84}$$

Here $\Box = \nabla^\mu\nabla_\mu$ is the covariant D'Alembertian, and it is crucial to realize when we take the covariant derivative that the four functions x^μ are just functions, not components of a vector. This condition is therefore simply

$$\begin{aligned} 0 &= \Box x^\mu \\ &= g^{\rho\sigma}\partial_\rho\partial_\sigma x^\mu - g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}\partial_\lambda x^\mu \\ &= -g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma} \ . \end{aligned} \tag{4.85}$$

In flat space, of course, Cartesian coordinates (in which $\Gamma^\lambda_{\rho\sigma} = 0$) are harmonic coordinates. (As a general principle, any function f which satisfies $\Box f = 0$ is called an "harmonic function.")

To see that this choice of coordinates successfully fixes our gauge freedom, let's rewrite the condition (4.84) in a somewhat simpler form. We have

$$g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\rho\sigma}g^{\mu\nu}(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}) \ , \tag{4.86}$$

from the definition of the Christoffel symbols. Meanwhile, from $\partial_\rho(g^{\mu\nu}g_{\sigma\nu}) = \partial_\rho\delta^\mu_\sigma = 0$ we have

$$g^{\mu\nu}\partial_\rho g_{\sigma\nu} = -g_{\sigma\nu}\partial_\rho g^{\mu\nu} \ . \tag{4.87}$$

Also, from our previous exploration of the variation of the determinant of the metric (4.65), we have

$$\frac{1}{2}g_{\rho\sigma}\partial_\nu g^{\rho\sigma} = -\frac{1}{\sqrt{-g}}\,\partial_\nu\sqrt{-g} \ . \tag{4.88}$$

Putting it all together, we find that (in general),

$$g^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma} = \frac{1}{\sqrt{-g}}\,\partial_\lambda(\sqrt{-g}g^{\lambda\mu}) \ . \tag{4.89}$$

The harmonic gauge condition (4.85) therefore is equivalent to

$$\partial_\lambda(\sqrt{-g}g^{\lambda\mu}) = 0 \ . \tag{4.90}$$

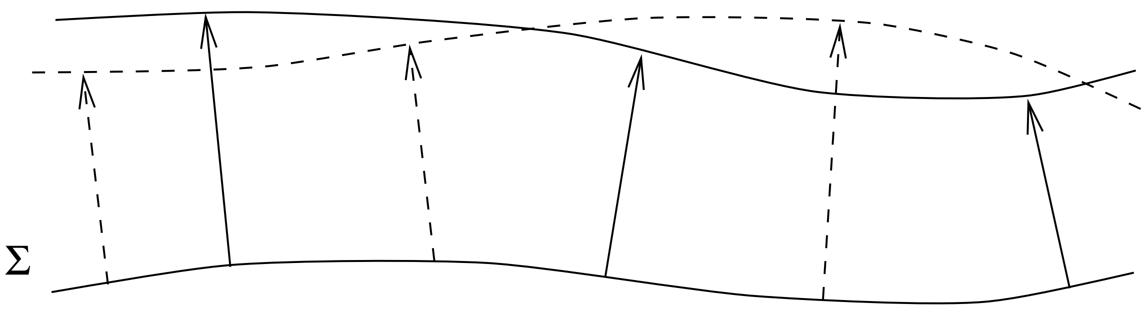
Taking the partial derivative of this with respect to $t = x^0$ yields

$$\frac{\partial^2}{\partial t^2}(\sqrt{-g}g^{0\nu}) = -\frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial t}(\sqrt{-g}g^{i\nu})\right] \ . \tag{4.91}$$

This condition represents a second-order differential equation for the previously unconstrained metric components $g^{0\nu}$, in terms of the given initial data. We have therefore succeeded in fixing our gauge freedom, in that we can now solve for the evolution of the entire metric in harmonic coordinates. (At least locally; we have been glossing over the fact our gauge choice may not be well-defined globally, and we would have to resort to working in patches as usual. The same problem appears in gauge theories in particle physics.) Note that we still have some freedom remaining; our gauge condition (4.84) restricts how the coordinates stretch from our initial hypersurface Σ throughout spacetime, but we can still choose coordinates x^i on Σ however we like. This corresponds to the fact that making a coordinate transformation $x^\mu \to x^\mu + \delta^\mu$, with $\Box\delta^\mu = 0$, does not violate the harmonic gauge condition.

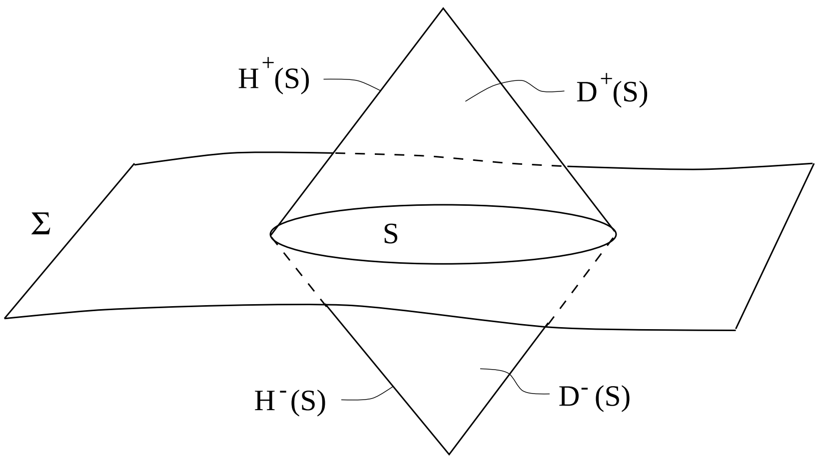
We therefore have a well-defined initial value problem for general relativity; a state is specified by the spacelike components of the metric and their time derivatives on a spacelike hypersurface Σ; given these, the spacelike components (4.83) of Einstein's equations allow us to evolve the metric forward in time, up to an ambiguity in coordinate choice which may be resolved by choice of gauge. We must keep in mind that the initial data are not arbitrary, but must obey the constraints (4.82). (Once we impose the constraints on some spacelike hypersurface, the equations of motion guarantee that they remain satisfied, as you can check.) The constraints serve a useful purpose, of guaranteeing that the result remains spacetime covariant after we have split our manifold into "space" and "time." Specifically, the $G^{i0} = 8\pi G T^{i0}$ constraint implies that the evolution is independent of our choice of coordinates on Σ, while $G^{00} = 8\pi G T^{00}$ enforces invariance under different ways of slicing spacetime into spacelike hypersurfaces.

Once we have seen how to cast Einstein's equations as an initial value problem, one issue of crucial importance is the existence of solutions to the problem. That is, once we have specified a spacelike hypersurface with initial data, to what extent can we be guaranteed that a unique spacetime will be determined? Although one can do a great deal of hard work



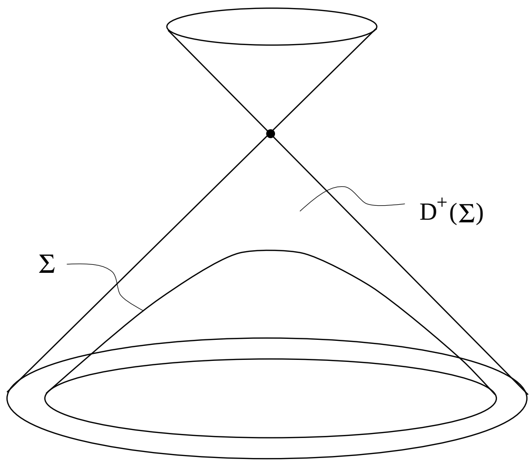
to answer this question with some precision, it is fairly simple to get a handle on the ways in which a well-defined solution can fail to exist, which we now consider.

It is simplest to first consider the problem of evolving matter fields on a fixed background spacetime, rather than the evolution of the metric itself. We therefore consider a spacelike hypersurface Σ in some manifold M with fixed metric $g_{\mu\nu}$, and furthermore look at some connected subset S in Σ. Our guiding principle will be that no signals can travel faster than the speed of light; therefore "information" will only flow along timelike or null trajectories (not necessarily geodesics). We define the **future domain of dependence** of S, denoted $D^+(S)$, as the set of all points p such that *every* past-moving, timelike or null, inextendible curve through p must intersect S. ("Inextendible" just means that the curve goes on forever, not ending at some finite point.) We interpret this definition in such a way that S itself is a subset of $D^+(S)$. (Of course a rigorous formulation does not require additional interpretation over and above the definitions, but we are not being as rigorous as we could be right now.) Similarly, we define the past domain of dependence $D^-(S)$ in the same way, but with "past-moving" replaced by "future-moving." Generally speaking, some points in M will be in one of the domains of dependence, and some will be outside; we define the boundary of $D^+(S)$ to be the **future Cauchy horizon** $H^+(S)$, and likewise the boundary of $D^-(S)$ to be the past Cauchy horizon $H^-(S)$. You can convince yourself that they are both null surfaces.



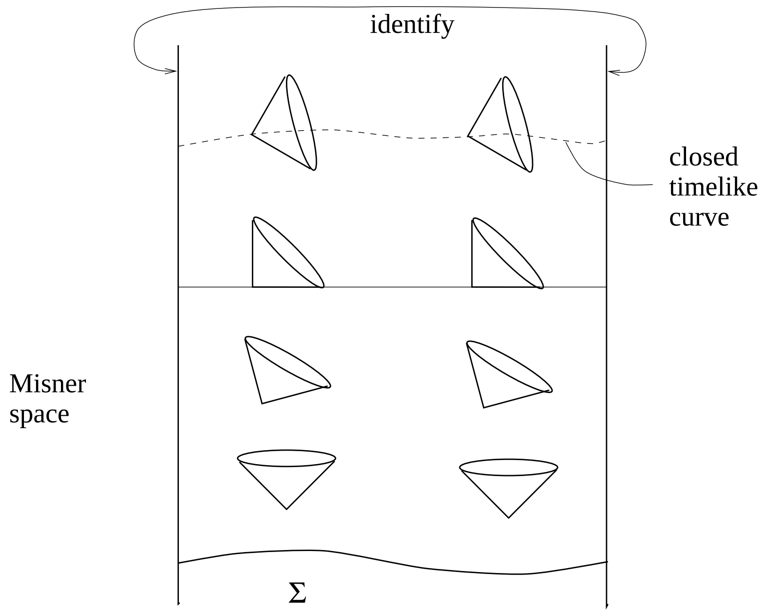
The usefulness of these definitions should be apparent; if nothing moves faster than light, than signals cannot propagate outside the light cone of any point p. Therefore, if every curve which remains inside this light cone must intersect S, then information specified on S should be sufficient to predict what the situation is at p. (That is, initial data for matter fields given on S can be used to solve for the value of the fields at p.) The set of all points for which we can predict what happens by knowing what happens on S is simply the union $D^+(S) \cup D^-(S)$.

We can easily extend these ideas from the subset S to the entire hypersurface Σ. The important point is that $D^+(\Sigma) \cup D^-(\Sigma)$ might fail to be all of M, even if Σ itself seems like a perfectly respectable hypersurface that extends throughout space. There are a number of ways in which this can happen. One possibility is that we have just chosen a "bad" hypersurface (although it is hard to give a general prescription for when a hypersurface is bad in this sense). Consider Minkowski space, and a spacelike hypersurface Σ which remains to the past of the light cone of some point.



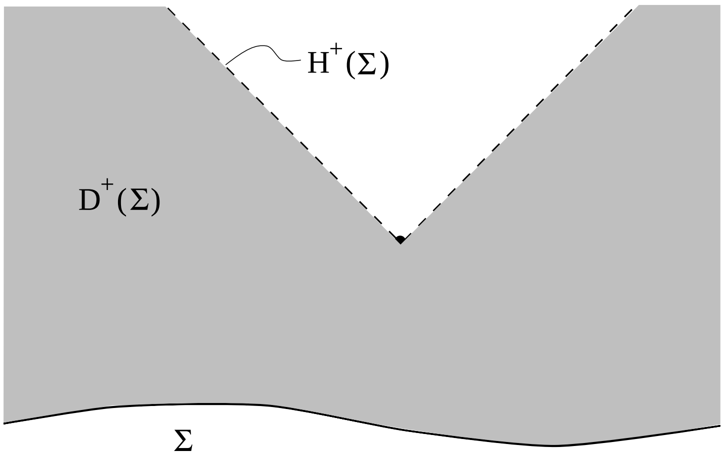
In this case Σ is a nice spacelike surface, but it is clear that $D^+(\Sigma)$ ends at the light cone, and we cannot use information on Σ to predict what happens throughout Minkowski space. Of course, there are other surfaces we could have picked for which the domain of dependence would have been the entire manifold, so this doesn't worry us too much.

A somewhat more nontrivial example is known as **Misner space**. This is a two-dimensional spacetime with the topology of $\mathbf{R}^1 \times S^1$, and a metric for which the light cones progressively tilt as you go forward in time. Past a certain point, it is possible to travel on a timelike trajectory which wraps around the S^1 and comes back to itself; this is known as a **closed timelike curve**. If we had specified a surface Σ to this past of this point, then none of the points in the region containing closed timelike curves are in the domain of dependence of Σ, since the closed timelike curves themselves do not intersect Σ. This is obviously a worse problem than the previous one, since a well-defined initial value problem does not seem to



exist in this spacetime. (Actually problems like this are the subject of some current research interest, so I won't claim that the issue is settled.)

A final example is provided by the existence of singularities, points which are not in the manifold even though they can be reached by travelling along a geodesic for a finite distance. Typically these occur when the curvature becomes infinite at some point; if this happens, the point can no longer be said to be part of the spacetime. Such an occurrence can lead to the emergence of a Cauchy horizon — a point p which is in the future of a singularity cannot be in the domain of dependence of a hypersurface to the past of the singularity, because there will be curves from p which simply end at the singularity.

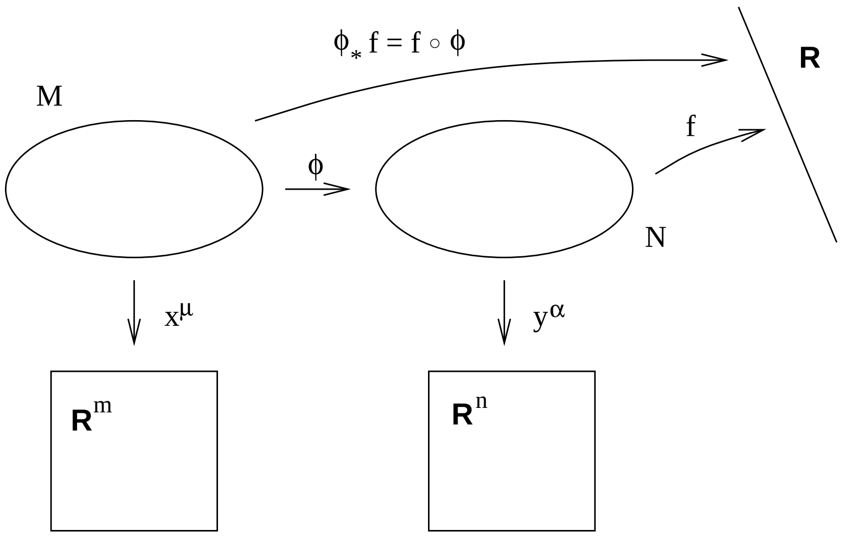


All of these obstacles can also arise in the initial value problem for GR, when we try to evolve the metric itself from initial data. However, they are of different degrees of troublesomeness. The possibility of picking a "bad" initial hypersurface does not arise very often, especially since most solutions are found globally (by solving Einstein's equations throughout spacetime). The one situation in which you have to be careful is in numerical solution of Einstein's equations, where a bad choice of hypersurface can lead to numerical difficulties even if in principle a complete solution exists. Closed timelike curves seem to be something that GR works hard to avoid — there are certainly solutions which contain them, but evolution from generic initial data does not usually produce them. Singularities, on the other hand, are practically unavoidable. The simple fact that the gravitational force is always attractive tends to pull matter together, increasing the curvature, and generally leading to some sort of singularity. This is something which we apparently must learn to live with, although there is some hope that a well-defined theory of quantum gravity will eliminate the singularities of classical GR.

5 More Geometry

With an understanding of how the laws of physics adapt to curved spacetime, it is undeniably tempting to start in on applications. However, a few extra mathematical techniques will simplify our task a great deal, so we will pause briefly to explore the geometry of manifolds some more.

When we discussed manifolds in section 2, we introduced maps between two different manifolds and how maps could be composed. We now turn to the use of such maps in carrying along tensor fields from one manifold to another. We therefore consider two manifolds M and N, possibly of different dimension, with coordinate systems x^μ and y^α, respectively. We imagine that we have a map $\phi : M \to N$ and a function $f : N \to \mathbf{R}$.



It is obvious that we can compose ϕ with f to construct a map $(f \circ \phi) : M \to \mathbf{R}$, which is simply a function on M. Such a construction is sufficiently useful that it gets its own name; we define the **pullback** of f by ϕ, denoted $\phi_* f$, by

$$\phi_* f = (f \circ \phi) . \tag{5.1}$$

The name makes sense, since we think of ϕ_* as "pulling back" the function f from N to M.

We can pull functions back, but we cannot push them forward. If we have a function $g : M \to \mathbf{R}$, there is no way we can compose g with ϕ to create a function on N; the arrows don't fit together correctly. But recall that a vector can be thought of as a derivative operator that maps smooth functions to real numbers. This allows us to define the **pushforward** of a vector; if $V(p)$ is a vector at a point p on M, we define the pushforward vector $\phi^* V$ at the point $\phi(p)$ on N by giving its action on functions on N:

$$(\phi^* V)(f) = V(\phi_* f) \ . \tag{5.2}$$

So to push forward a vector field we say "the action of $\phi^* V$ on any function is simply the action of V on the pullback of that function."

This is a little abstract, and it would be nice to have a more concrete description. We know that a basis for vectors on M is given by the set of partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$, and a basis on N is given by the set of partial derivatives $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$. Therefore we would like to relate the components of $V = V^\mu \partial_\mu$ to those of $(\phi^* V) = (\phi^* V)^\alpha \partial_\alpha$. We can find the sought-after relation by applying the pushed-forward vector to a test function and using the chain rule (2.3):

$$\begin{aligned}(\phi^* V)^\alpha \partial_\alpha f &= V^\mu \partial_\mu (\phi_* f) \\ &= V^\mu \partial_\mu (f \circ \phi) \\ &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f \ .\end{aligned} \tag{5.3}$$

This simple formula makes it irresistible to think of the pushforward operation ϕ^* as a matrix operator, $(\phi^* V)^\alpha = (\phi^*)^\alpha{}_\mu V^\mu$, with the matrix being given by

$$(\phi^*)^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \ . \tag{5.4}$$

The behavior of a vector under a pushforward thus bears an unmistakable resemblance to the vector transformation law under change of coordinates. In fact it is a generalization, since when M and N are the same manifold the constructions are (as we shall discuss) identical; but don't be fooled, since in general μ and α have different allowed values, and there is no reason for the matrix $\partial y^\alpha / \partial x^\mu$ to be invertible.

It is a rewarding exercise to convince yourself that, although you can push vectors forward from M to N (given a map $\phi : M \to N$), you cannot in general pull them back — just keep trying to invent an appropriate construction until the futility of the attempt becomes clear. Since one-forms are dual to vectors, you should not be surprised to hear that one-forms can be pulled back (but not in general pushed forward). To do this, remember that one-forms are linear maps from vectors to the real numbers. The pullback $\phi_* \omega$ of a one-form ω on N can therefore be defined by its action on a vector V on M, by equating it with the action of ω itself on the pushforward of V:

$$(\phi_* \omega)(V) = \omega(\phi^* V) \ . \tag{5.5}$$

Once again, there is a simple matrix description of the pullback operator on forms, $(\phi_* \omega)_\mu = (\phi_*)_\mu{}^\alpha \omega_\alpha$, which we can derive using the chain rule. It is given by

$$(\phi_*)_\mu{}^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} \ . \tag{5.6}$$

That is, it is the same matrix as the pushforward (5.4), but of course a different index is contracted when the matrix acts to pull back one-forms.

There is a way of thinking about why pullbacks and pushforwards work on some objects but not others, which may or may not be helpful. If we denote the set of smooth functions on M by $\mathcal{F}(M)$, then a vector $V(p)$ at a point p on M (*i.e.*, an element of the tangent space T_pM) can be thought of as an operator from $\mathcal{F}(M)$ to $\mathbf{R}$. But we already know that the pullback operator on functions maps $\mathcal{F}(N)$ to $\mathcal{F}(M)$ (just as ϕ itself maps M to N, but in the opposite direction). Therefore we can define the pushforward ϕ_* acting on vectors simply by composing maps, as we first defined the pullback of functions:

$$\begin{tikzcd} \mathbf{R} & \\ \mathcal{F}(M) \arrow[u, "V(p)"] & \mathcal{F}(N) \arrow[l, "\phi_*"'] \arrow[ul, "\phi^*(V(p)) = V(p) \circ \phi_*"'] \end{tikzcd}$$

Similarly, if T_qN is the tangent space at a point q on N, then a one-form ω at q (*i.e.*, an element of the cotangent space T_q^*N) can be thought of as an operator from T_qN to $\mathbf{R}$. Since the pushforward ϕ^* maps T_pM to $T_{\phi(p)}N$, the pullback ϕ_* of a one-form can also be thought of as mere composition of maps:

$$\begin{tikzcd} & \mathbf{R} \\ T_pM \arrow[ur, "\phi_*(\omega) = \omega \circ \phi^*"] \arrow[r, "\phi^*"] & T_{\phi(p)}N \arrow[u, "\omega"'] \end{tikzcd}$$

If this is not helpful, don't worry about it. But do keep straight what exists and what doesn't; the actual concepts are simple, it's just remembering which map goes what way that leads to confusion.

You will recall further that a $(0, l)$ tensor — one with l lower indices and no upper ones — is a linear map from the direct product of l vectors to $\mathbf{R}$. We can therefore pull back not only one-forms, but tensors with an arbitrary number of lower indices. The definition is simply the action of the original tensor on the pushed-forward vectors:

$$(\phi_* T)(V^{(1)}, V^{(2)}, \ldots, V^{(l)}) = T(\phi^* V^{(1)}, \phi^* V^{(2)}, \ldots, \phi^* V^{(l)}) \ , \tag{5.7}$$

where $T_{\alpha_1 \cdots \alpha_l}$ is a $(0, l)$ tensor on N. We can similarly push forward any $(k, 0)$ tensor $S^{\mu_1 \cdots \mu_k}$ by acting it on pulled-back one-forms:

$$(\phi^* S)(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(k)}) = S(\phi_* \omega^{(1)}, \phi_* \omega^{(2)}, \ldots, \phi_* \omega^{(k)}) \ . \tag{5.8}$$

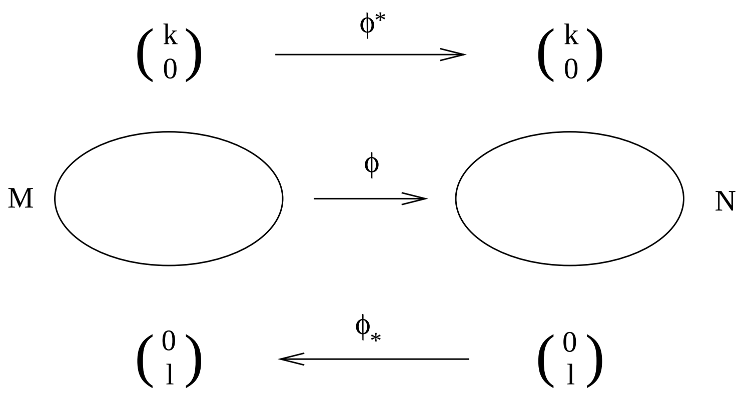
Fortunately, the matrix representations of the pushforward (5.4) and pullback (5.6) extend to the higher-rank tensors simply by assigning one matrix to each index; thus, for the pullback of a $(0, l)$ tensor, we have

$$(\phi_* T)_{\mu_1 \cdots \mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1 \cdots \alpha_l} \ , \tag{5.9}$$

while for the pushforward of a $(k, 0)$ tensor we have

$$(\phi^* S)^{\alpha_1 \cdots \alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1 \cdots \mu_k} \ . \tag{5.10}$$

Our complete picture is therefore:



Note that tensors with both upper and lower indices can generally be neither pushed forward nor pulled back.

This machinery becomes somewhat less imposing once we see it at work in a simple example. One common occurrence of a map between two manifolds is when M is actually a submanifold of N; then there is an obvious map from M to N which just takes an element of M to the "same" element of N. Consider our usual example, the two-sphere embedded in $\mathbf{R}^3$, as the locus of points a unit distance from the origin. If we put coordinates $x^\mu = (\theta, \phi)$ on $M = S^2$ and $y^\alpha = (x, y, z)$ on $N = \mathbf{R}^3$, the map $\phi : M \to N$ is given by

$$\phi(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \ . \tag{5.11}$$

In the past we have considered the metric $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbf{R}^3$, and said that it induces a metric $\mathrm{d}\theta^2 + \sin^2\theta \ \mathrm{d}\phi^2$ on S^2, just by substituting (5.11) into this flat metric on

$\mathbf{R}^3$. We didn't really justify such a statement at the time, but now we can do so. (Of course it would be easier if we worked in spherical coordinates on $\mathbf{R}^3$, but doing it the hard way is more illustrative.) The matrix of partial derivatives is given by

$$\frac{\partial y^\alpha}{\partial x^\mu} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} . \tag{5.12}$$

The metric on S^2 is obtained by simply pulling back the metric from $\mathbf{R}^3$,

$$\begin{aligned} (\phi^* g)_{\mu\nu} &= \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} , \end{aligned} \tag{5.13}$$

as you can easily check. Once again, the answer is the same as you would get by naive substitution, but now we know why.

We have been careful to emphasize that a map $\phi : M \to N$ can be used to push certain things forward and pull other things back. The reason why it generally doesn't work both ways can be traced to the fact that ϕ might not be invertible. If ϕ is invertible (and both ϕ and ϕ^{-1} are smooth, which we always implicitly assume), then it defines a diffeomorphism between M and N. In this case M and N are the same abstract manifold. The beauty of diffeomorphisms is that we can use both ϕ and ϕ^{-1} to move tensors from M to N; this will allow us to define the pushforward and pullback of arbitrary tensors. Specifically, for a (k, l) tensor field $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}$ on M, we define the pushforward by

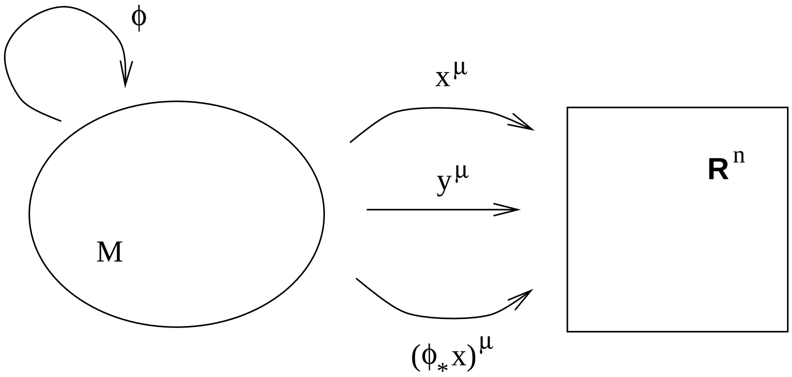
$$(\phi^* T)(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T(\phi_*\omega^{(1)}, \ldots, \phi_*\omega^{(k)}, [\phi^{-1}]^* V^{(1)}, \ldots, [\phi^{-1}]^* V^{(l)}) , \tag{5.14}$$

where the $\omega^{(i)}$'s are one-forms on N and the $V^{(i)}$'s are vectors on N. In components this becomes

$$(\phi^* T)^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} . \tag{5.15}$$

The appearance of the inverse matrix $\partial x^\nu / \partial y^\beta$ is legitimate because ϕ is invertible. Note that we could also define the pullback in the obvious way, but there is no need to write separate equations because the pullback ϕ_* is the same as the pushforward via the inverse map, $[\phi^{-1}]^*$.

We are now in a position to explain the relationship between diffeomorphisms and coordinate transformations. The relationship is that they are two different ways of doing precisely the same thing. If you like, diffeomorphisms are "active coordinate transformations", while traditional coordinate transformations are "passive." Consider an n-dimensional manifold M with coordinate functions $x^\mu : M \to \mathbf{R}^n$. To change coordinates we can either simply introduce new functions $y^\mu : M \to \mathbf{R}^n$ ("keep the manifold fixed, change the coordinate maps"), or we could just as well introduce a diffeomorphism $\phi : M \to M$, after which the coordinates would just be the pullbacks $(\phi_* x)^\mu : M \to \mathbf{R}^n$ ("move the points on the manifold, and then evaluate the coordinates of the new points"). In this sense, (5.15) really is the tensor transformation law, just thought of from a different point of view.



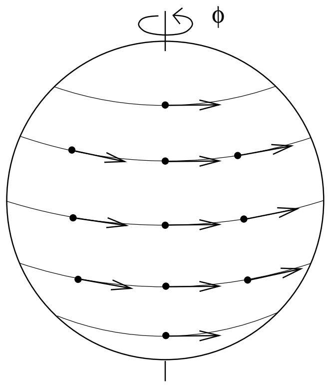
Since a diffeomorphism allows us to pull back and push forward arbitrary tensors, it provides another way of comparing tensors at different points on a manifold. Given a diffeomorphism $\phi : M \to M$ and a tensor field $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \mu_l}(x)$, we can form the difference between the value of the tensor at some point p and $\phi_*[T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \mu_l}(\phi(p))]$, its value at $\phi(p)$ pulled back to p. This suggests that we could define another kind of derivative operator on tensor fields, one which categorizes the rate of change of the tensor as it changes under the diffeomorphism. For that, however, a single discrete diffeomorphism is insufficient; we require a one-parameter family of diffeomorphisms, ϕ_t. This family can be thought of as a smooth map $\mathbf{R} \times M \to M$, such that for each $t \in \mathbf{R}$ ϕ_t is a diffeomorphism and $\phi_s \circ \phi_t = \phi_{s+t}$. Note that this last condition implies that ϕ_0 is the identity map.

One-parameter families of diffeomorphisms can be thought of as arising from vector fields (and vice-versa). If we consider what happens to the point p under the entire family ϕ_t, it is clear that it describes a curve in M; since the same thing will be true of every point on M, these curves fill the manifold (although there can be degeneracies where the diffeomorphisms have fixed points). We can define a vector field $V^\mu(x)$ to be the set of tangent vectors to each of these curves at every point, evaluated at $t = 0$. An example on S^2 is provided by the diffeomorphism $\phi_t(\theta, \phi) = (\theta, \phi + t)$.

We can reverse the construction to define a one-parameter family of diffeomorphisms from any vector field. Given a vector field $V^\mu(x)$, we define the **integral curves** of the vector field to be those curves $x^\mu(t)$ which solve

$$\frac{dx^\mu}{dt} = V^\mu \ . \tag{5.16}$$

Note that this familiar-looking equation is now to be interpreted in the opposite sense from our usual way — we are given the vectors, from which we define the curves. Solutions to

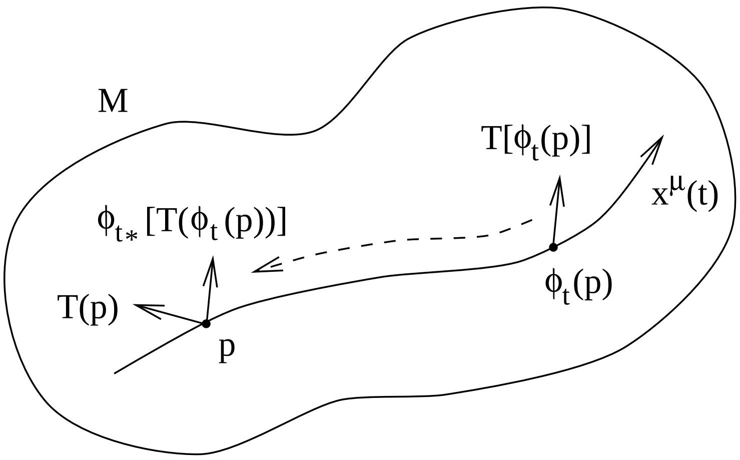


(5.16) are guaranteed to exist as long as we don't do anything silly like run into the edge of our manifold; any standard differential geometry text will have the proof, which amounts to finding a clever coordinate system in which the problem reduces to the fundamental theorem of ordinary differential equations. Our diffeomorphisms ϕ_t represent "flow down the integral curves," and the associated vector field is referred to as the **generator** of the diffeomorphism. (Integral curves are used all the time in elementary physics, just not given the name. The "lines of magnetic flux" traced out by iron filings in the presence of a magnet are simply the integral curves of the magnetic field vector **B**.)

Given a vector field $V^\mu(x)$, then, we have a family of diffeomorphisms parameterized by t, and we can ask how fast a tensor changes as we travel down the integral curves. For each t we can define this change as

$$\Delta_t T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}(p) = \phi_{t*}[T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}(\phi_t(p))] - T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}(p) \ . \tag{5.17}$$

Note that both terms on the right hand side are tensors at p.



We then define the **Lie derivative** of the tensor along the vector field as

$$\pounds_V T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l} = \lim_{t\to 0}\left(\frac{\Delta_t T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}}{t}\right) \ . \tag{5.18}$$

The Lie derivative is a map from (k, l) tensor fields to (k, l) tensor fields, which is manifestly independent of coordinates. Since the definition essentially amounts to the conventional definition of an ordinary derivative applied to the component functions of the tensor, it should be clear that it is linear,

$$\pounds_V(aT + bS) = a\pounds_V T + b\pounds_V S \ , \tag{5.19}$$

and obeys the Leibniz rule,

$$\pounds_V(T \otimes S) = (\pounds_V T) \otimes S + T \otimes (\pounds_V S) \ , \tag{5.20}$$

where S and T are tensors and a and b are constants. The Lie derivative is in fact a more primitive notion than the covariant derivative, since it does not require specification of a connection (although it does require a vector field, of course). A moment's reflection shows that it reduces to the ordinary derivative on functions,

$$\pounds_V f = V(f) = V^\mu \partial_\mu f \ . \tag{5.21}$$

To discuss the action of the Lie derivative on tensors in terms of other operations we know, it is convenient to choose a coordinate system adapted to our problem. Specifically, we will work in coordinates x^μ for which x^1 is the parameter along the integral curves (and the other coordinates are chosen any way we like). Then the vector field takes the form $V = \partial/\partial x^1$; that is, it has components $V^\mu = (1, 0, 0, \ldots, 0)$. The magic of this coordinate system is that a diffeomorphism by t amounts to a coordinate transformation from x^μ to $y^\mu = (x^1 + t, x^2, \ldots, x^n)$. Thus, from (5.6) the pullback matrix is simply

$$(\phi_{t*})_\mu{}^\nu = \delta^\nu_\mu \ , \tag{5.22}$$

and the components of the tensor pulled back from $\phi_t(p)$ to p are simply

$$\phi_{t*}[T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}(\phi_t(p))] = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l}(x^1 + t, x^2, \ldots, x^n) \ . \tag{5.23}$$

In this coordinate system, then, the Lie derivative becomes

$$\pounds_V T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l} = \frac{\partial}{\partial x^1} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\mu_l} \ , \tag{5.24}$$

and specifically the derivative of a vector field $U^\mu(x)$ is

$$\pounds_V U^\mu = \frac{\partial U^\mu}{\partial x^1} \ . \tag{5.25}$$

Although this expression is clearly not covariant, we know that the commutator $[V, U]$ is a well-defined tensor, and in this coordinate system

$$[V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu$$

$$= \frac{\partial U^\mu}{\partial x^1} \; . \tag{5.26}$$

Therefore the Lie derivative of U with respect to V has the same components in this coordinate system as the commutator of V and U; but since both are vectors, they must be equal in any coordinate system:

$$\pounds_V U^\mu = [V, U]^\mu \; . \tag{5.27}$$

As an immediate consequence, we have $\pounds_V S = -\pounds_W V$. It is because of (5.27) that the commutator is sometimes called the "Lie bracket."

To derive the action of $\pounds_V$ on a one-form ω_μ, begin by considering the action on the scalar $\omega_\mu U^\mu$ for an arbitrary vector field U^μ. First use the fact that the Lie derivative with respect to a vector field reduces to the action of the vector itself when applied to a scalar:

$$\begin{aligned} \pounds_V(\omega_\mu U^\mu) &= V(\omega_\mu U^\mu) \\ &= V^\nu \partial_\nu(\omega_\mu U^\mu) \\ &= V^\nu(\partial_\nu \omega_\mu) U^\mu + V^\nu \omega_\mu (\partial_\nu U^\mu) \; . \end{aligned} \tag{5.28}$$

Then use the Leibniz rule on the original scalar:

$$\begin{aligned} \pounds_V(\omega_\mu U^\mu) &= (\pounds_V \omega)_\mu U^\mu + \omega_\mu (\pounds_V U)^\mu \\ &= (\pounds_V \omega)_\mu U^\mu + \omega_\mu V^\nu \partial_\nu U^\mu - \omega_\mu U^\nu \partial_\nu V^\mu \; . \end{aligned} \tag{5.29}$$

Setting these expressions equal to each other and requiring that equality hold for arbitrary U^μ, we see that

$$\pounds_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu \; , \tag{5.30}$$

which (like the definition of the commutator) is completely covariant, although not manifestly so.

By a similar procedure we can define the Lie derivative of an arbitrary tensor field. The answer can be written

$$\begin{aligned} \pounds_V T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= V^\sigma \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &\quad -(\partial_\lambda V^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - (\partial_\lambda V^{\mu_2}) T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - \cdots \\ &\quad +(\partial_{\nu_1} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} + (\partial_{\nu_2} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} + \cdots \end{aligned} \tag{5.31}$$

Once again, this expression is covariant, despite appearances. It would undoubtedly be comforting, however, to have an equivalent expression that looked manifestly tensorial. In fact it turns out that we can write

$$\begin{aligned} \pounds_V T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= V^\sigma \nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &\quad -(\nabla_\lambda V^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - (\nabla_\lambda V^{\mu_2}) T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - \cdots \\ &\quad +(\nabla_{\nu_1} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} + (\nabla_{\nu_2} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} + \cdots \end{aligned} \tag{5.32}$$

where ∇_μ represents *any* symmetric (torsion-free) covariant derivative (including, of course, one derived from a metric). You can check that all of the terms which would involve connection coefficients if we were to expand (5.32) would cancel, leaving only (5.31). Both versions of the formula for a Lie derivative are useful at different times. A particularly useful formula is for the Lie derivative of the metric:

$$\begin{aligned} \pounds_V g_{\mu\nu} &= V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu \\ &= 2\nabla_{(\mu} V_{\nu)} \ , \end{aligned} \tag{5.33}$$

where ∇_μ is the covariant derivative derived from $g_{\mu\nu}$.

Let's put some of these ideas into the context of general relativity. You will often hear it proclaimed that GR is a "diffeomorphism invariant" theory. What this means is that, if the universe is represented by a manifold M with metric $g_{\mu\nu}$ and matter fields ψ, and $\phi : M \to M$ is a diffeomorphism, then the sets $(M, g_{\mu\nu}, \psi)$ and $(M, \phi_* g_{\mu\nu}, \phi_* \psi)$ represent the same physical situation. Since diffeomorphisms are just active coordinate transformations, this is a highbrow way of saying that the theory is coordinate invariant. Although such a statement is true, it is a source of great misunderstanding, for the simple fact that it conveys very little information. Any semi-respectable theory of physics is coordinate invariant, including those based on special relativity or Newtonian mechanics; GR is not unique in this regard. When people say that GR is diffeomorphism invariant, more likely than not they have one of two (closely related) concepts in mind: the theory is free of "prior geometry", and there is no preferred coordinate system for spacetime. The first of these stems from the fact that the metric is a dynamical variable, and along with it the connection and volume element and so forth. Nothing is given to us ahead of time, unlike in classical mechanics or SR. As a consequence, there is no way to simplify life by sticking to a specific coordinate system adapted to some absolute elements of the geometry. This state of affairs forces us to be very careful; it is possible that two purportedly distinct configurations (of matter and metric) in GR are actually "the same", related by a diffeomorphism. In a path integral approach to quantum gravity, where we would like to sum over all possible configurations, special care must be taken not to overcount by allowing physically indistinguishable configurations to contribute more than once. In SR or Newtonian mechanics, meanwhile, the existence of a preferred set of coordinates saves us from such ambiguities. The fact that GR has no *preferred* coordinate system is often garbled into the statement that it is coordinate invariant (or "generally covariant"); both things are true, but one has more content than the other.

On the other hand, the fact of diffeomorphism invariance can be put to good use. Recall that the complete action for gravity coupled to a set of matter fields ψ^i is given by a sum of the Hilbert action for GR plus the matter action,

$$S = \frac{1}{8\pi G} S_H[g_{\mu\nu}] + S_M[g_{\mu\nu}, \psi^i] \ . \tag{5.34}$$

The Hilbert action S_H is diffeomorphism invariant when considered in isolation, so the matter action S_M must also be if the action as a whole is to be invariant. We can write the variation in S_M under a diffeomorphism as

$$\delta S_M = \int d^n x \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int d^n x \frac{\delta S_M}{\delta \psi^i} \delta \psi^i \ . \tag{5.35}$$

We are not considering arbitrary variations of the fields, only those which result from a diffeomorphism. Nevertheless, the matter equations of motion tell us that the variation of S_M with respect to ψ^i will vanish for any variation (since the gravitational part of the action doesn't involve the matter fields). Hence, for a diffeomorphism invariant theory the first term on the right hand side of (5.35) must vanish. If the diffeomorphism in generated by a vector field $V^\mu(x)$, the infinitesimal change in the metric is simply given by its Lie derivative along V^μ; by (5.33) we have

$$\begin{aligned} \delta g_{\mu\nu} &= \pounds_V g_{\mu\nu} \\ &= 2\nabla_{(\mu} V_{\nu)} \ . \end{aligned} \tag{5.36}$$

Setting $\delta S_M = 0$ then implies

$$\begin{aligned} 0 &= \int d^n x \frac{\delta S_M}{\delta g_{\mu\nu}} \nabla_\mu V_\nu \\ &= -\int d^n x \sqrt{-g} V_\nu \nabla_\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right) \ , \end{aligned} \tag{5.37}$$

where we are able to drop the symmetrization of $\nabla_{(\mu} V_{\nu)}$ since $\delta S_M / \delta g_{\mu\nu}$ is already symmetric. Demanding that (5.37) hold for diffeomorphisms generated by arbitrary vector fields V^μ, and using the definition (4.70) of the energy-momentum tensor, we obtain precisely the law of energy-momentum conservation,

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{5.38}$$

This is why we claimed earlier that the conservation of $T_{\mu\nu}$ was more than simply a consequence of the Principle of Equivalence; it is much more secure than that, resting only on the diffeomorphism invariance of the theory.

There is one more use to which we will put the machinery we have set up in this section: symmetries of tensors. We say that a diffeomorphism ϕ is a **symmetry** of some tensor T if the tensor is invariant after being pulled back under ϕ:

$$\phi_* T = T \ . \tag{5.39}$$

Although symmetries may be discrete, it is more common to have a one-parameter family of symmetries ϕ_t. If the family is generated by a vector field $V^\mu(x)$, then (5.39) amounts to

$$\pounds_V T = 0 \ . \tag{5.40}$$

By (5.25), one implication of a symmetry is that, if T is symmetric under some one-parameter family of diffeomorphisms, we can always find a coordinate system in which the components of T are all independent of one of the coordinates (the integral curve coordinate of the vector field). The converse is also true; if all of the components are independent of one of the coordinates, then the partial derivative vector field associated with that coordinate generates a symmetry of the tensor.

The most important symmetries are those of the metric, for which $\phi_* g_{\mu\nu} = g_{\mu\nu}$. A diffeomorphism of this type is called an **isometry**. If a one-parameter family of isometries is generated by a vector field $V^\mu(x)$, then V^μ is known as a **Killing vector field**. The condition that V^μ be a Killing vector is thus

$$\pounds_V g_{\mu\nu} = 0 \ , \tag{5.41}$$

or from (5.33),

$$\nabla_{(\mu} V_{\nu)} = 0 \ . \tag{5.42}$$

This last version is **Killing's equation**. If a spacetime has a Killing vector, then we know we can find a coordinate system in which the metric is independent of one of the coordinates.

By far the most useful fact about Killing vectors is that *Killing vectors imply conserved quantities associated with the motion of free particles.* If $x^\mu(\lambda)$ is a geodesic with tangent vector $U^\mu = dx^\mu/d\lambda$, and V^μ is a Killing vector, then

$$\begin{aligned} U^\nu \nabla_\nu (V_\mu U^\mu) &= U^\nu U^\mu \nabla_\nu V_\mu + V_\mu U^\nu \nabla_\nu U^\mu \\ &= 0 \ , \end{aligned} \tag{5.43}$$

where the first term vanishes from Killing's equation and the second from the fact that $x^\mu(\lambda)$ is a geodesic. Thus, the quantity $V_\mu U^\mu$ is conserved along the particle's worldline. This can be understood physically: by definition the metric is unchanging along the direction of the Killing vector. Loosely speaking, therefore, a free particle will not feel any "forces" in this direction, and the component of its momentum in that direction will consequently be conserved.

Long ago we referred to the concept of a space with maximal symmetry, without offering a rigorous definition. The rigorous definition is that a **maximally symmetric space** is one which possesses the largest possible number of Killing vectors, which on an n-dimensional manifold is $n(n+1)/2$. We will not prove this statement, but it is easy to understand at an informal level. Consider the Euclidean space $\mathbf{R}^n$, where the isometries are well known to us: translations and rotations. In general there will be n translations, one for each direction we can move. There will also be $n(n-1)/2$ rotations; for each of n dimensions there are $n-1$ directions in which we can rotate it, but we must divide by two to prevent overcounting (rotating x into y and rotating y into x are two versions of the same thing). We therefore have

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \tag{5.44}$$

independent Killing vectors. The same kind of counting argument applies to maximally symmetric spaces with curvature (such as spheres) or a non-Euclidean signature (such as Minkowski space), although the details are marginally different.

Although it may or may not be simple to actually solve Killing's equation in any given spacetime, it is frequently possible to write down some Killing vectors by inspection. (Of course a "generic" metric has no Killing vectors at all, but to keep things simple we often deal with metrics with high degrees of symmetry.) For example in $\mathbf{R}^2$ with metric $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, independence of the metric components with respect to x and y immediately yields two Killing vectors:

$$\begin{aligned} X^\mu &= (1, 0) , \\ Y^\mu &= (0, 1) . \end{aligned} \tag{5.45}$$

These clearly represent the two translations. The one rotation would correspond to the vector $R = \partial/\partial\theta$ if we were in polar coordinates; in Cartesian coordinates this becomes

$$R^\mu = (-y, x) . \tag{5.46}$$

You can check for yourself that this actually does solve Killing's equation.

Note that in $n \geq 2$ dimensions, there can be more Killing vectors than dimensions. This is because a set of Killing vector fields can be linearly independent, even though at any one point on the manifold the vectors at that point are linearly dependent. It is trivial to show (so you should do it yourself) that a linear combination of Killing vectors with *constant* coefficients is still a Killing vector (in which case the linear combination does not count as an independent Killing vector), but this is not necessarily true with coefficients which vary over the manifold. You will also show that the commutator of two Killing vector fields is a Killing vector field; this is very useful to know, but it may be the case that the commutator gives you a vector field which is not linearly independent (or it may simply vanish). The problem of finding all of the Killing vectors of a metric is therefore somewhat tricky, as it is sometimes not clear when to stop looking.

6 Weak Fields and Gravitational Radiation

When we first derived Einstein's equations, we checked that we were on the right track by considering the Newtonian limit. This amounted to the requirements that the gravitational field be weak, that it be static (no time derivatives), and that test particles be moving slowly. In this section we will consider a less restrictive situation, in which the field is still weak but it can vary with time, and there are no restrictions on the motion of test particles. This will allow us to discuss phenomena which are absent or ambiguous in the Newtonian theory, such as gravitational radiation (where the field varies with time) and the deflection of light (which involves fast-moving particles).

The weakness of the gravitational field is once again expressed as our ability to decompose the metric into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| << 1 \ . \tag{6.1}$$

We will restrict ourselves to coordinates in which $\eta_{\mu\nu}$ takes its canonical form, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. The assumption that $h_{\mu\nu}$ is small allows us to ignore anything that is higher than first order in this quantity, from which we immediately obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{6.2}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. As before, we can raise and lower indices using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, since the corrections would be of higher order in the perturbation. In fact, we can think of the linearized version of general relativity (where effects of higher than first order in $h_{\mu\nu}$ are neglected) as describing a theory of a symmetric tensor field $h_{\mu\nu}$ propagating on a flat background spacetime. This theory is Lorentz invariant in the sense of special relativity; under a Lorentz transformation $x^{\mu'} = \Lambda^{\mu'}{}_{\mu}x^{\mu}$, the flat metric $\eta_{\mu\nu}$ is invariant, while the perturbation transforms as

$$h_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}h_{\mu\nu} \ . \tag{6.3}$$

(Note that we could have considered small perturbations about some other background spacetime besides Minkowski space. In that case the metric would have been written $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, and we would have derived a theory of a symmetric tensor propagating on the curved space with metric $g^{(0)}_{\mu\nu}$. Such an approach is necessary, for example, in cosmology.)

We want to find the equation of motion obeyed by the perturbations $h_{\mu\nu}$, which come by examining Einstein's equations to first order. We begin with the Christoffel symbols, which are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$= \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \ . \tag{6.4}$$

Since the connection coefficients are first order quantities, the only contribution to the Riemann tensor will come from the derivatives of the Γ's, not the Γ^2 terms. Lowering an index for convenience, we obtain

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho} \\ &= \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}) \ . \end{aligned} \tag{6.5}$$

The Ricci tensor comes from contracting over μ and ρ, giving

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \ , \tag{6.6}$$

which is manifestly symmetric in μ and ν. In this expression we have defined the trace of the perturbation as $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu{}_\mu$, and the D'Alembertian is simply the one from flat space, $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. Contracting again to obtain the Ricci scalar yields

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \ . \tag{6.7}$$

Putting it all together we obtain the Einstein tensor:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h) \ . \end{aligned} \tag{6.8}$$

Consistent with our interpretation of the linearized theory as one describing a symmetric tensor on a flat background, the linearized Einstein tensor (6.8) can be derived by varying the following Lagrangian with respect to $h_{\mu\nu}$:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right] \ . \tag{6.9}$$

I will spare you the details.

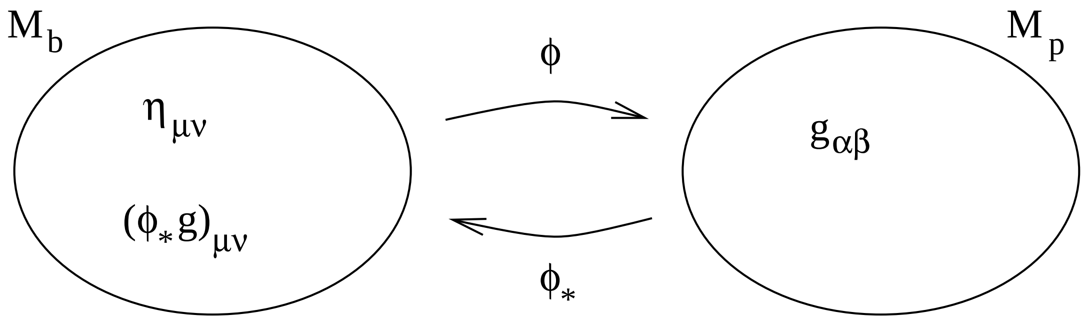
The linearized field equation is of course $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ is given by (6.8) and $T_{\mu\nu}$ is the energy-momentum tensor, calculated to zeroth order in $h_{\mu\nu}$. We do not include higher-order corrections to the energy-momentum tensor because the amount of energy and momentum must itself be small for the weak-field limit to apply. In other words, the lowest nonvanishing order in $T_{\mu\nu}$ is automatically of the same order of magnitude as the perturbation. Notice that the conservation law to lowest order is simply $\partial_\mu T^{\mu\nu} = 0$. We will most often be concerned with the vacuum equations, which as usual are just $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is given by (6.6).

With the linearized field equations in hand, we are almost prepared to set about solving them. First, however, we should deal with the thorny issue of gauge invariance. This issue arises because the demand that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ does not completely specify the coordinate system on spacetime; there may be other coordinate systems in which the metric can still be written as the Minkowski metric plus a small perturbation, but the perturbation will be different. Thus, the decomposition of the metric into a flat background plus a perturbation is not unique.

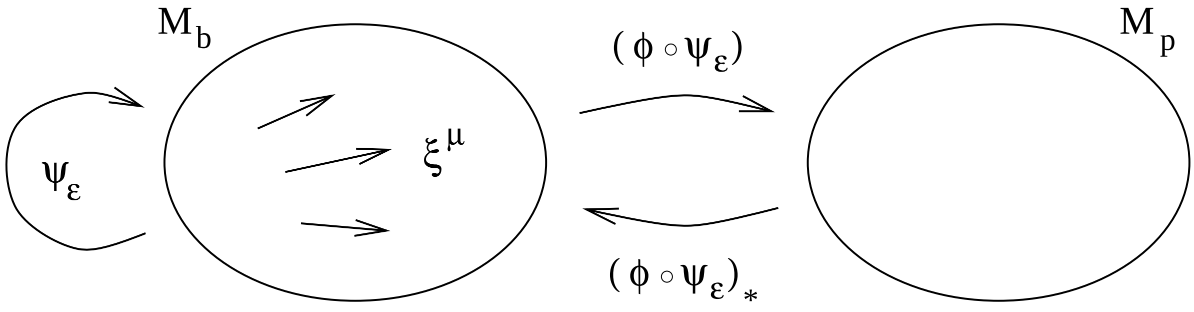
We can think about this from a highbrow point of view. The notion that the linearized theory can be thought of as one governing the behavior of tensor fields on a flat background can be formalized in terms of a "background spacetime" M_b, a "physical spacetime" M_p, and a diffeomorphism $\phi : M_b \rightarrow M_p$. As manifolds M_b and M_p are "the same" (since they are diffeomorphic), but we imagine that they possess some different tensor fields; on M_b we have defined the flat Minkowski metric $\eta_{\mu\nu}$, while on M_p we have some metric $g_{\alpha\beta}$ which obeys Einstein's equations. (We imagine that M_b is equipped with coordinates x^μ and M_p is equipped with coordinates y^α, although these will not play a prominent role.) The diffeomorphism ϕ allows us to move tensors back and forth between the background and physical spacetimes. Since we would like to construct our linearized theory as one taking place on the flat background spacetime, we are interested in the pullback $(\phi_* g)_{\mu\nu}$ of the physical metric. We can define the perturbation as the difference between the pulled-back physical metric and the flat one:

$$h_{\mu\nu} = (\phi_* g)_{\mu\nu} - \eta_{\mu\nu} \ . \tag{6.10}$$

From this definition, there is no reason for the components of $h_{\mu\nu}$ to be small; however, if the gravitational fields on M_p are weak, then for *some* diffeomorphisms ϕ we will have $|h_{\mu\nu}| << 1$. We therefore limit our attention only to those diffeomorphisms for which this is true. Then the fact that $g_{\alpha\beta}$ obeys Einstein's equations on the physical spacetime means that $h_{\mu\nu}$ will obey the linearized equations on the background spacetime (since ϕ, as a diffeomorphism, can be used to pull back Einstein's equations themselves).



In this language, the issue of gauge invariance is simply the fact that there are a large number of permissible diffeomorphisms between M_b and M_p (where "permissible" means that the perturbation is small). Consider a vector field $\xi^\mu(x)$ on the background spacetime. This vector field generates a one-parameter family of diffeomorphisms $\psi_\epsilon : M_b \to M_b$. For ϵ sufficiently small, if ϕ is a diffeomorphism for which the perturbation defined by (6.10) is small than so will $(\phi \circ \psi_\epsilon)$ be, although the perturbation will have a different value.



Specifically, we can define a family of perturbations parameterized by ϵ:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ \psi_\epsilon)_* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\psi_{\epsilon *}(\phi_* g)]_{\mu\nu} - \eta_{\mu\nu} \ . \end{aligned} \tag{6.11}$$

The second equality is based on the fact that the pullback under a composition is given by the composition of the pullbacks in the opposite order, which follows from the fact that the pullback itself moves things in the opposite direction from the original map. Plugging in the relation (6.10), we find

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= \psi_{\epsilon *}(h+\eta)_{\mu\nu} - \eta_{\mu\nu} \\ &= \psi_{\epsilon *}(h_{\mu\nu}) + \psi_{\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu} \end{aligned} \tag{6.12}$$

(since the pullback of the sum of two tensors is the sum of the pullbacks). Now we use our assumption that ϵ is small; in this case $\psi_{\epsilon *}(h_{\mu\nu})$ will be equal to $h_{\mu\nu}$ to lowest order, while the other two terms give us a Lie derivative:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= \psi_{\epsilon *}(h_{\mu\nu}) + \epsilon\left[\frac{\psi_{\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon}\right] \\ &= h_{\mu\nu} + \epsilon \pounds_\xi \eta_{\mu\nu} \\ &= h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)} \ . \end{aligned} \tag{6.13}$$

The last equality follows from our previous computation of the Lie derivative of the metric, (5.33), plus the fact that covariant derivatives are simply partial derivatives to lowest order.

The infinitesimal diffeomorphisms ϕ_ϵ provide a different representation of the same physical situation, while maintaining our requirement that the perturbation be small. Therefore, the result (6.12) tells us what kind of metric perturbations denote physically equivalent spacetimes — those related to each other by $2\epsilon\partial_{(\mu}\xi_{\nu)}$, for some vector ξ^μ. The invariance of our theory under such transformations is analogous to traditional gauge invariance of electromagnetism under $A_\mu \to A_\mu + \partial_\mu\lambda$. (The analogy is different from the previous analogy we drew with electromagnetism, relating local Lorentz transformations in the orthonormal-frame formalism to changes of basis in an internal vector bundle.) In electromagnetism the invariance comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by gauge transformations; similarly, we find that the transformation (6.13) changes the linearized Riemann tensor by

$$\begin{aligned} \delta R_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu \\ &\qquad -\partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\sigma\partial_\nu\partial_\rho\xi_\mu - \partial_\rho\partial_\mu\partial_\nu\xi_\sigma - \partial_\rho\partial_\mu\partial_\sigma\xi_\nu) \\ &= 0\ . \end{aligned} \tag{6.14}$$

Our abstract derivation of the appropriate gauge transformation for the metric perturbation is verified by the fact that it leaves the curvature (and hence the physical spacetime) unchanged.

Gauge invariance can also be understood from the slightly more lowbrow but considerably more direct route of infinitesimal coordinate transformations. Our diffeomorphism ψ_ϵ can be thought of as changing coordinates from x^μ to $x^\mu - \epsilon\xi^\mu$. (The minus sign, which is unconventional, comes from the fact that the "new" metric is pulled back from a small distance forward along the integral curves, which is equivalent to replacing the coordinates by those a small distance backward along the curves.) Following through the usual rules for transforming tensors under coordinate transformations, you can derive precisely (6.13) — although you have to cheat somewhat by equating components of tensors in two different coordinate systems. See Schutz or Weinberg for an example.

When faced with a system that is invariant under some kind of gauge transformations, our first instinct is to fix a gauge. We have already discussed the harmonic coordinate system, and will return to it now in the context of the weak field limit. Recall that this gauge was specified by $\Box x^\mu = 0$, which we showed was equivalent to

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0\ . \tag{6.15}$$

In the weak field limit this becomes

$$\frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) = 0\ , \tag{6.16}$$

or

$$\partial_\mu h^\mu{}_\lambda - \frac{1}{2}\partial_\lambda h = 0\ . \tag{6.17}$$

This condition is also known as Lorentz gauge (or Einstein gauge or Hilbert gauge or de Donder gauge or Fock gauge). As before, we still have some gauge freedom remaining, since we can change our coordinates by (infinitesimal) harmonic functions.

In this gauge, the linearized Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ simplify somewhat, to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu} \ , \tag{6.18}$$

while the vacuum equations $R_{\mu\nu} = 0$ take on the elegant form

$$\Box h_{\mu\nu} = 0 \ , \tag{6.19}$$

which is simply the conventional relativistic wave equation. Together, (6.19) and (6.17) determine the evolution of a disturbance in the gravitational field in vacuum in the harmonic gauge.

It is often convenient to work with a slightly different description of the metric perturbation. We define the "trace-reversed" perturbation $\bar{h}_{\mu\nu}$ by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \ . \tag{6.20}$$

The name makes sense, since $\bar{h}^\mu{}_\mu = -h^\mu{}_\mu$. (The Einstein tensor is simply the trace-reversed Ricci tensor.) In terms of $\bar{h}_{\mu\nu}$ the harmonic gauge condition becomes

$$\partial_\mu \bar{h}^\mu{}_\lambda = 0 \ . \tag{6.21}$$

The full field equations are

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \tag{6.22}$$

from which it follows immediately that the vacuum equations are

$$\Box \bar{h}_{\mu\nu} = 0 \ . \tag{6.23}$$

From (6.22) and our previous exploration of the Newtonian limit, it is straightforward to derive the weak-field metric for a stationary spherical source such as a planet or star. Recall that previously we found that Einstein's equations predicted that h_{00} obeyed the Poisson equation (4.51) in the weak-field limit, which implied

$$h_{00} = -2\Phi \ , \tag{6.24}$$

where Φ is the conventional Newtonian potential, $\Phi = -GM/r$. Let us now assume that the energy-momentum tensor of our source is dominated by its rest energy density $\rho = T_{00}$. (Such an assumption is not generally necessary in the weak-field limit, but will certainly hold for a planet or star, which is what we would like to consider for the moment.) Then the other components of $T_{\mu\nu}$ will be much smaller than T_{00}, and from (6.22) the same must hold for $\bar{h}_{\mu\nu}$. If $\bar{h}_{00}$ is much larger than $\bar{h}_{ij}$, we will have

$$h = -\bar{h} = -\eta^{\mu\nu}\bar{h}_{\mu\nu} = \bar{h}_{00} \ , \tag{6.25}$$

and then from (6.20) we immediately obtain

$$\bar{h}_{00} = 2h_{00} = -4\Phi \ . \tag{6.26}$$

The other components of $\bar{h}_{\mu\nu}$ are negligible, from which we can derive

$$h_{i0} = \bar{h}_{i0} - \frac{1}{2}\eta_{i0}\bar{h} = 0 \ , \tag{6.27}$$

and

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} = -2\Phi\delta_{ij} \ . \tag{6.28}$$

The metric for a star or planet in the weak-field limit is therefore

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \ . \tag{6.29}$$

A somewhat less simplistic application of the weak-field limit is to gravitational radiation. Those of you familiar with the analogous problem in electromagnetism will notice that the procedure is almost precisely the same. We begin by considering the linearized equations in vacuum (6.23). Since the flat-space D'Alembertian has the form $\Box = -\partial_t^2 + \nabla^2$, the field equation is in the form of a wave equation for $\bar{h}_{\mu\nu}$. As all good physicists know, the thing to do when faced with such an equation is to write down complex-valued solutions, and then take the real part at the very end of the day. So we recognize that a particularly useful set of solutions to this wave equation are the plane waves, given by

$$\bar{h}_{\mu\nu} = C_{\mu\nu}e^{ik_\sigma x^\sigma} \ , \tag{6.30}$$

where $C_{\mu\nu}$ is a constant, symmetric, $(0,2)$ tensor, and k^σ is a constant vector known as the **wave vector**. To check that it is a solution, we plug in:

$$\begin{aligned} 0 &= \Box\bar{h}_{\mu\nu} \\ &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma\bar{h}_{\mu\nu} \\ &= \eta^{\rho\sigma}\partial_\rho(ik_\sigma\bar{h}_{\mu\nu}) \\ &= -\eta^{\rho\sigma}k_\rho k_\sigma\bar{h}_{\mu\nu} \\ &= -k_\sigma k^\sigma\bar{h}_{\mu\nu} \ . \end{aligned} \tag{6.31}$$

Since (for an interesting solution) not all of the components of $h_{\mu\nu}$ will be zero everywhere, we must have

$$k_\sigma k^\sigma = 0 \ . \tag{6.32}$$

The plane wave (6.30) is therefore a solution to the linearized equations if the wavevector is null; this is loosely translated into the statement that gravitational waves propagate at the speed of light. The timelike component of the wave vector is often referred to as the

frequency of the wave, and we write $k^\sigma = (\omega, k^1, k^2, k^3)$. (More generally, an observer moving with four-velocity U^μ would observe the wave to have a frequency $\omega = -k_\mu U^\mu$.) Then the condition that the wave vector be null becomes

$$\omega^2 = \delta_{ij} k^i k^j \ . \tag{6.33}$$

Of course our wave is far from the most general solution; any (possibly infinite) number of distinct plane waves can be added together and will still solve the linear equation (6.23). Indeed, any solution can be written as such a superposition.

There are a number of free parameters to specify the wave: ten numbers for the coefficients $C_{\mu\nu}$ and three for the null vector k^σ. Much of these are the result of coordinate freedom and gauge freedom, which we now set about eliminating. We begin by imposing the harmonic gauge condition, (6.21). This implies that

$$\begin{aligned} 0 &= \partial_\mu \bar{h}^{\mu\nu} \\ &= \partial_\mu (C^{\mu\nu} e^{ik_\sigma x^\sigma}) \\ &= iC^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma} \ , \end{aligned} \tag{6.34}$$

which is only true if

$$k_\mu C^{\mu\nu} = 0 \ . \tag{6.35}$$

We say that the wave vector is orthogonal to $C^{\mu\nu}$. These are four equations, which reduce the number of independent components of $C_{\mu\nu}$ from ten to six.

Although we have now imposed the harmonic gauge condition, there is still some coordinate freedom left. Remember that any coordinate transformation of the form

$$x^\mu \to x^\mu + \zeta^\mu \tag{6.36}$$

will leave the harmonic coordinate condition

$$\Box x^\mu = 0 \tag{6.37}$$

satisfied as long as we have

$$\Box \zeta^\mu = 0 \ . \tag{6.38}$$

Of course, (6.38) is itself a wave equation for ζ^μ; once we choose a solution, we will have used up all of our gauge freedom. Let's choose the following solution:

$$\zeta_\mu = B_\mu e^{ik_\sigma x^\sigma} \ , \tag{6.39}$$

where k_σ is the wave vector for our gravitational wave and the B_μ are constant coefficients.

We now claim that this remaining freedom allows us to convert from whatever coefficients $C^{(\mathrm{old})}_{\mu\nu}$ that characterize our gravitational wave to a new set $C^{(\mathrm{new})}_{\mu\nu}$, such that

$$C^{(\mathrm{new})\mu}{}_\mu = 0 \tag{6.40}$$

and

$$C^{(\mathrm{new})}_{0\nu} = 0 \ . \tag{6.41}$$

(Actually this last condition is both a choice of gauge and a choice of Lorentz frame. The choice of gauge sets $U^\mu C^{(\mathrm{new})}_{\mu\nu} = 0$ for some constant timelike vector U^μ, while the choice of frame makes U^μ point along the time axis.) Let's see how this is possible by solving explicitly for the necessary coefficients B_μ. Under the transformation (6.36), the resulting change in our metric perturbation can be written

$$h^{(\mathrm{new})}_{\mu\nu} = h^{(\mathrm{old})}_{\mu\nu} - \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu \ , \tag{6.42}$$

which induces a change in the trace-reversed perturbation,

$$\begin{aligned}
\bar{h}^{(\mathrm{new})}_{\mu\nu} &= h^{(\mathrm{new})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(\mathrm{new})} \\
&= h^{(\mathrm{old})}_{\mu\nu} - \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu - \frac{1}{2}\eta_{\mu\nu}(h^{(\mathrm{old})} - 2\partial_\lambda\zeta^\lambda) \\
&= \bar{h}^{(\mathrm{old})}_{\mu\nu} - \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu + \eta_{\mu\nu}\partial_\lambda\zeta^\lambda \ .
\end{aligned} \tag{6.43}$$

Using the specific forms (6.30) for the solution and (6.39) for the transformation, we obtain

$$C^{(\mathrm{new})}_{\mu\nu} = C^{(\mathrm{old})}_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu}k_\lambda B^\lambda \ . \tag{6.44}$$

Imposing (6.40) therefore means

$$0 = C^{(\mathrm{old})\mu}{}_\mu + 2ik_\lambda B^\lambda \ , \tag{6.45}$$

or

$$k_\lambda B^\lambda = \frac{i}{2}C^{(\mathrm{old})\mu}{}_\mu \ . \tag{6.46}$$

Then we can impose (6.41), first for $\nu = 0$:

$$\begin{aligned}
0 &= C^{(\mathrm{old})}_{00} - 2ik_0B_0 - ik_\lambda B^\lambda \\
&= C^{(\mathrm{old})}_{00} - 2ik_0B_0 + \frac{1}{2}C^{(\mathrm{old})\mu}{}_\mu \ ,
\end{aligned} \tag{6.47}$$

or

$$B_0 = -\frac{i}{2k_0}\left(C^{(\mathrm{old})}_{00} + \frac{1}{2}C^{(\mathrm{old})\mu}{}_\mu\right) \ . \tag{6.48}$$

Then impose (6.41) for $\nu = j$:

$$\begin{aligned}
0 &= C^{(\mathrm{old})}_{0j} - ik_0B_j - ik_jB_0 \\
&= C^{(\mathrm{old})}_{0j} - ik_0B_j - ik_j\left[\frac{-i}{2k_0}\left(C^{(\mathrm{old})}_{00} + \frac{1}{2}C^{(\mathrm{old})\mu}{}_\mu\right)\right] \ ,
\end{aligned} \tag{6.49}$$

or

$$B_j = \frac{i}{2(k_0)^2}\left[-2k_0 C^{(\mathrm{old})}_{0j} + k_j\left(C^{(\mathrm{old})}_{00} + \frac{1}{2}C^{(\mathrm{old})\mu}{}_{\mu}\right)\right] . \tag{6.50}$$

To check that these choices are mutually consistent, we should plug (6.48) and (6.50) back into (6.40), which I will leave to you. Let us assume that we have performed this transformation, and refer to the new components $C^{(\mathrm{new})}_{\mu\nu}$ simply as $C_{\mu\nu}$.

Thus, we began with the ten independent numbers in the symmetric matrix $C_{\mu\nu}$. Choosing harmonic gauge implied the four conditions (6.35), which brought the number of independent components down to six. Using our remaining gauge freedom led to the one condition (6.40) and the four conditions (6.41); but when $\nu = 0$ (6.41) implies (6.35), so we have a total of four additional constraints, which brings us to two independent components. We've used up all of our possible freedom, so these two numbers represent the physical information characterizing our plane wave in this gauge. This can be seen more explicitly by choosing our spatial coordinates such that the wave is travelling in the x^3 direction; that is,

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega) , \tag{6.51}$$

where we know that $k^3 = \omega$ because the wave vector is null. In this case, $k^\mu C_{\mu\nu} = 0$ and $C_{0\nu} = 0$ together imply

$$C_{3\nu} = 0 . \tag{6.52}$$

The only nonzero components of $C_{\mu\nu}$ are therefore C_{11}, C_{12}, C_{21}, and C_{22}. But $C_{\mu\nu}$ is traceless and symmetric, so in general we can write

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.53}$$

Thus, for a plane wave in this gauge travelling in the x^3 direction, the two components C_{11} and C_{12} (along with the frequency ω) completely characterize the wave.

In using up all of our gauge freedom, we have gone to a subgauge of the harmonic gauge known as the **transverse traceless gauge** (or sometimes "radiation gauge"). The name comes from the fact that the metric perturbation is traceless and perpendicular to the wave vector. Of course, we have been working with the trace-reversed perturbation $\bar{h}_{\mu\nu}$ rather than the perturbation $h_{\mu\nu}$ itself; but since $\bar{h}_{\mu\nu}$ is traceless (because $C_{\mu\nu}$ is), and is equal to the trace-reverse of $h_{\mu\nu}$, in this gauge we have

$$\bar{h}^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} \qquad \text{(transverse traceless gauge)} . \tag{6.54}$$

Therefore we can drop the bars over $h_{\mu\nu}$, as long as we are in this gauge.

One nice feature of the transverse traceless gauge is that if you are given the components of a plane wave in some arbitrary gauge, you can easily convert them into the transverse traceless components. We first define a tensor $P_{\mu\nu}$ which acts as a projection operator:

$$P_{\mu\nu} = \eta_{\mu\nu} - n_\mu n_\nu \ . \tag{6.55}$$

You can check that this projects vectors onto hyperplanes orthogonal to the unit vector n_μ. Here we take n_μ to be a *spacelike* unit vector, which we choose to lie along the direction of propagation of the wave:

$$n_0 = 0 \ , \qquad n_j = k_j/\omega \ . \tag{6.56}$$

Then the transverse part of some perturbation $h_{\mu\nu}$ is simply the projection $P_\mu{}^\rho P_\nu{}^\sigma h_{\rho\sigma}$, and the transverse traceless part is obtained by subtracting off the trace:

$$h^{\rm TT}_{\mu\nu} = P_\mu{}^\rho P_\nu{}^\sigma h_{\rho\sigma} - \frac{1}{2} P_{\mu\nu} P^{\rho\sigma} h_{\rho\sigma} \ . \tag{6.57}$$

For details appropriate to more general cases, see the discussion in Misner, Thorne and Wheeler.

To get a feeling for the physical effects due to gravitational waves, it is useful to consider the motion of test particles in the presence of a wave. It is certainly insufficient to solve for the trajectory of a single particle, since that would only tell us about the values of the coordinates along the world line. (In fact, for any single particle we can find transverse traceless coordinates in which the particle appears stationary to first order in $h_{\mu\nu}$.) To obtain a coordinate-independent measure of the wave's effects, we consider the relative motion of nearby particles, as described by the geodesic deviation equation. If we consider some nearby particles with four-velocities described by a single vector field $U^\mu(x)$ and separation vector S^μ, we have

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma \ . \tag{6.58}$$

We would like to compute the left-hand side to first order in $h_{\mu\nu}$. If we take our test particles to be moving slowly then we can express the four-velocity as a unit vector in the time direction plus corrections of order $h_{\mu\nu}$ and higher; but we know that the Riemann tensor is already first order, so the corrections to U^ν may be ignored, and we write

$$U^\nu = (1, 0, 0, 0) \ . \tag{6.59}$$

Therefore we only need to compute $R^\mu{}_{00\sigma}$, or equivalently $R_{\mu 00\sigma}$. From (6.5) we have

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0\partial_0 h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{00} - \partial_\sigma\partial_0 h_{\mu 0} - \partial_\mu\partial_0 h_{\sigma 0}) \ . \tag{6.60}$$

But $h_{\mu 0} = 0$, so

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h_{\mu\sigma} \ . \tag{6.61}$$

Meanwhile, for our slowly-moving particles we have $\tau = x^0 = t$ to lowest order, so the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2}S^\mu = \frac{1}{2}S^\sigma \frac{\partial^2}{\partial t^2}h^\mu{}_\sigma \ . \tag{6.62}$$

For our wave travelling in the x^3 direction, this implies that only S^1 and S^2 will be affected — the test particles are only disturbed in directions perpendicular to the wave vector. This is of course familiar from electromagnetism, where the electric and magnetic fields in a plane wave are perpendicular to the wave vector.

Our wave is characterized by the two numbers, which for future convenience we will rename as $C_+ = C_{11}$ and $C_\times = C_{12}$. Let's consider their effects separately, beginning with the case $C_\times = 0$. Then we have

$$\frac{\partial^2}{\partial t^2}S^1 = \frac{1}{2}S^1 \frac{\partial^2}{\partial t^2}(C_+ e^{ik_\sigma x^\sigma}) \tag{6.63}$$

and

$$\frac{\partial^2}{\partial t^2}S^2 = -\frac{1}{2}S^2 \frac{\partial^2}{\partial t^2}(C_+ e^{ik_\sigma x^\sigma}) \ . \tag{6.64}$$

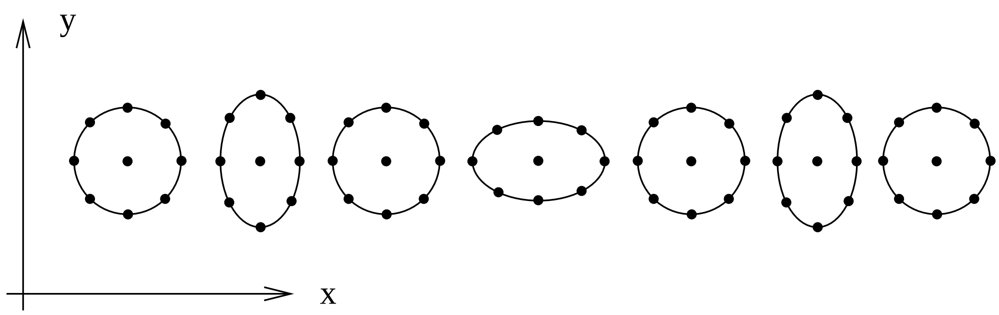
These can be immediately solved to yield, to lowest order,

$$S^1 = \left(1 + \frac{1}{2}C_+ e^{ik_\sigma x^\sigma}\right) S^1(0) \tag{6.65}$$

and

$$S^2 = \left(1 - \frac{1}{2}C_+ e^{ik_\sigma x^\sigma}\right) S^2(0) \ . \tag{6.66}$$

Thus, particles initially separated in the x^1 direction will oscillate back and forth in the x^1 direction, and likewise for those with an initial x^2 separation. That is, if we start with a ring of stationary particles in the x-y plane, as the wave passes they will bounce back and forth in the shape of a "+":



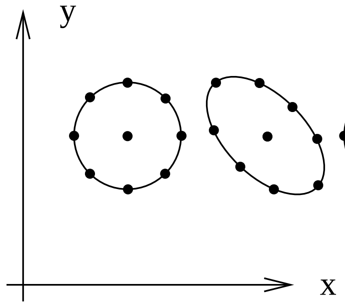
On the other hand, the equivalent analysis for the case where $C_+ = 0$ but $C_\times \neq 0$ would yield the solution

$$S^1 = S^1(0) + \frac{1}{2}C_\times e^{ik_\sigma x^\sigma} S^2(0) \tag{6.67}$$

and

$$S^2 = S^2(0) + \frac{1}{2}C_\times e^{ik_\sigma x^\sigma} S^1(0) \ . \tag{6.68}$$

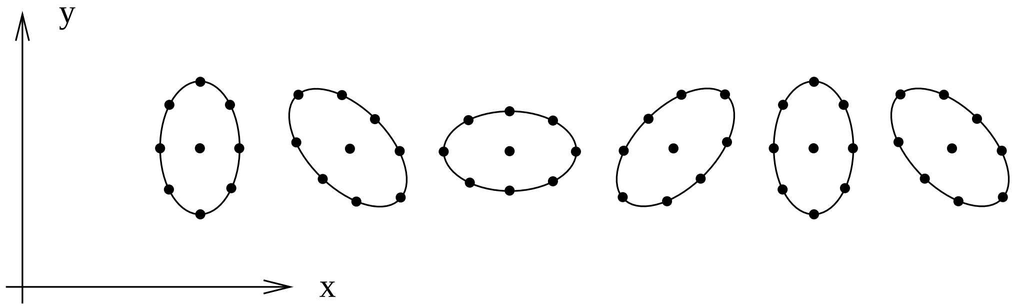
In this case the circle of particles would bounce back and forth in the shape of a "×":



The notation C_+ and $C_\times$ should therefore be clear. These two quantities measure the two independent modes of linear polarization of the gravitational wave. If we liked we could consider right- and left-handed circularly polarized modes by defining

$$\begin{aligned} C_R &= \frac{1}{\sqrt{2}}(C_+ + iC_\times) \ , \\ C_L &= \frac{1}{\sqrt{2}}(C_+ - iC_\times) \ . \end{aligned} \tag{6.69}$$

The effect of a pure C_R wave would be to rotate the particles in a right-handed sense,



and similarly for the left-handed mode C_L. (Note that the individual particles do not travel around the ring; they just move in little epicycles.)

We can relate the polarization states of classical gravitational waves to the kinds of particles we would expect to find upon quantization. The electromagnetic field has two independent polarization states which are described by vectors in the x-y plane; equivalently, a single polarization mode is invariant under a rotation by 360° in this plane. Upon quantization this theory yields the photon, a massless spin-one particle. The neutrino, on the other hand, is also a massless particle, described by a field which picks up a minus sign under rotations by 360°; it is invariant under rotations of 720°, and we say it has spin-$\frac{1}{2}$.

The general rule is that the spin S is related to the angle θ under which the polarization modes are invariant by $S = 360^\circ/\theta$. The gravitational field, whose waves propagate at the speed of light, should lead to massless particles in the quantum theory. Noticing that the polarization modes we have described are invariant under rotations of 180° in the x-y plane, we expect the associated particles — "gravitons" — to be spin-2. We are a long way from detecting such particles (and it would not be a surprise if we never detected them directly), but any respectable quantum theory of gravity should predict their existence.

With plane-wave solutions to the linearized vacuum equations in our possession, it remains to discuss the generation of gravitational radiation by sources. For this purpose it is necessary to consider the equations coupled to matter,

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ . \tag{6.70}$$

The solution to such an equation can be obtained using a Green's function, in precisely the same way as the analogous problem in electromagnetism. Here we will review the outline of the method.

The Green's function $G(x^\sigma - y^\sigma)$ for the D'Alembertian operator $\Box$ is the solution of the wave equation in the presence of a delta-function source:

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) \ , \tag{6.71}$$

where $\Box_x$ denotes the D'Alembertian with respect to the coordinates x^σ. The usefulness of such a function resides in the fact that the general solution to an equation such as (6.70) can be written

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) \ d^4y \ , \tag{6.72}$$

as can be verified immediately. (Notice that no factors of $\sqrt{-g}$ are necessary, since our background is simply flat spacetime.) The solutions to (6.71) have of course been worked out long ago, and they can be thought of as either "retarded" or "advanced," depending on whether they represent waves travelling forward or backward in time. Our interest is in the retarded Green's function, which represents the accumulated effects of signals to the past of the point under consideration. It is given by

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\delta[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)] \ \theta(x^0 - y^0) \ . \tag{6.73}$$

Here we have used boldface to denote the spatial vectors $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$, with norm $|\mathbf{x} - \mathbf{y}| = [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2}$. The theta function $\theta(x^0 - y^0)$ equals 1 when $x^0 > y^0$, and zero otherwise. The derivation of (6.73) would take us too far afield, but it can be found in any standard text on electrodynamics or partial differential equations in physics.

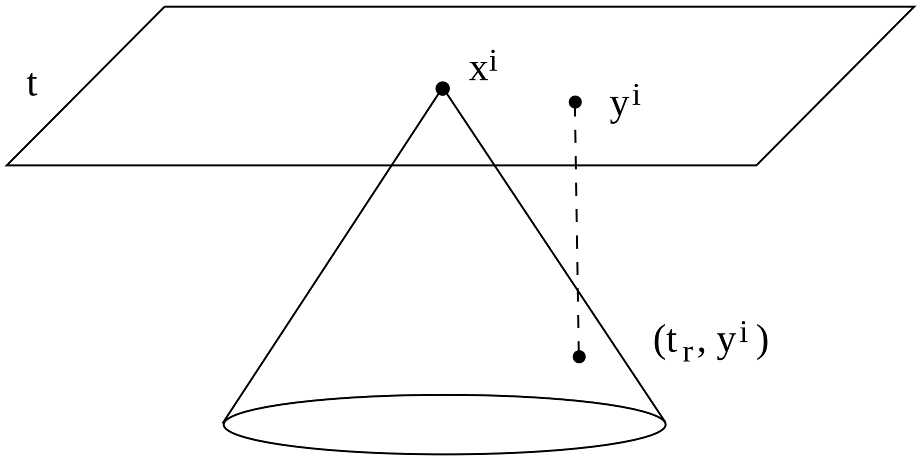
Upon plugging (6.73) into (6.72), we can use the delta function to perform the integral over y^0, leaving us with

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G\int \frac{1}{|\mathbf{x}-\mathbf{y}|}T_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})\ d^3y\ , \tag{6.74}$$

where $t = x^0$. The term "retarded time" is used to refer to the quantity

$$t_r = t - |\mathbf{x}-\mathbf{y}|\ . \tag{6.75}$$

The interpretation of (6.74) should be clear: the disturbance in the gravitational field at $(t,\mathbf{x})$ is a sum of the influences from the energy and momentum sources at the point $(t_r,\mathbf{x}-\mathbf{y})$ on the past light cone.



Let us take this general solution and consider the case where the gravitational radiation is emitted by an isolated source, fairly far away, comprised of nonrelativistic matter; these approximations will be made more precise as we go on. First we need to set up some conventions for Fourier transforms, which always make life easier when dealing with oscillatory phenomena. Given a function of spacetime $\phi(t,\mathbf{x})$, we are interested in its Fourier transform (and inverse) with respect to time alone,

$$\begin{aligned}\widetilde{\phi}(\omega,\mathbf{x}) &= \frac{1}{\sqrt{2\pi}}\int dt\ e^{i\omega t}\phi(t,\mathbf{x})\ ,\\ \phi(t,\mathbf{x}) &= \frac{1}{\sqrt{2\pi}}\int d\omega\ e^{-i\omega t}\widetilde{\phi}(\omega,\mathbf{x})\ .\end{aligned} \tag{6.76}$$

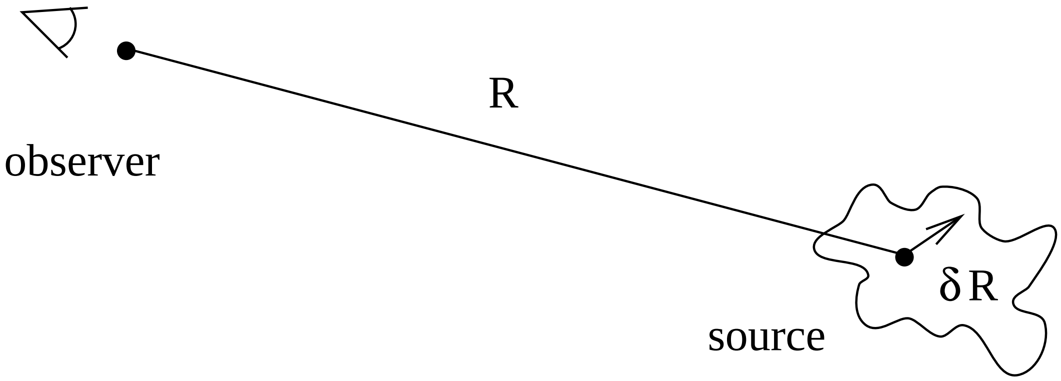
Taking the transform of the metric perturbation, we obtain

$$\widetilde{\bar{h}}_{\mu\nu}(\omega,\mathbf{x}) = \frac{1}{\sqrt{2\pi}}\int dt\ e^{i\omega t}\bar{h}_{\mu\nu}(t,\mathbf{x})$$

$$\begin{aligned}
&= \frac{4G}{\sqrt{2\pi}} \int dt\ d^3y\ e^{i\omega t}\ \frac{T_{\mu\nu}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\
&= \frac{4G}{\sqrt{2\pi}} \int dt_r\ d^3y\ e^{i\omega t_r} e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{T_{\mu\nu}(t_r, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\
&= 4G \int d^3y\ e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{\widetilde{T}_{\mu\nu}(\omega, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\ . \qquad (6.77)
\end{aligned}$$

In this sequence, the first equation is simply the definition of the Fourier transform, the second line comes from the solution (6.74), the third line is a change of variables from t to t_r, and the fourth line is once again the definition of the Fourier transform.

We now make the approximations that our source is isolated, far away, and slowly moving. This means that we can consider the source to be centered at a (spatial) distance R, with the different parts of the source at distances $R + \delta R$ such that $\delta R << R$. Since it is slowly moving, most of the radiation emitted will be at frequencies ω sufficiently low that $\delta R << \omega^{-1}$. (Essentially, light traverses the source much faster than the components of the source itself do.)



Under these approximations, the term $e^{i\omega|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ can be replaced by $e^{i\omega R}/R$ and brought outside the integral. This leaves us with

$$\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) = 4G\frac{e^{i\omega R}}{R} \int d^3y\ \widetilde{T}_{\mu\nu}(\omega, \mathbf{y})\ . \qquad (6.78)$$

In fact there is no need to compute all of the components of $\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x})$, since the harmonic gauge condition $\partial_\mu \bar{h}^{\mu\nu}(t, \mathbf{x}) = 0$ in Fourier space implies

$$\widetilde{\bar{h}}^{0\nu} = \frac{i}{\omega}\partial_i \widetilde{\bar{h}}^{i\nu}\ . \qquad (6.79)$$

We therefore only need to concern ourselves with the spacelike components of $\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x})$. From (6.78) we therefore want to take the integral of the spacelike components of $\widetilde{T}_{\mu\nu}(\omega, \mathbf{y})$.

We begin by integrating by parts in reverse:

$$\int d^3y\ \widetilde{T}^{ij}(\omega,\mathbf{y}) = \int \partial_k(y^i\widetilde{T}^{kj})\ d^3y - \int y^i(\partial_k\widetilde{T}^{kj})\ d^3y\ . \tag{6.80}$$

The first term is a surface integral which will vanish since the source is isolated, while the second can be related to $\widetilde{T}^{0j}$ by the Fourier-space version of $\partial_\mu T^{\mu\nu} = 0$:

$$-\partial_k\widetilde{T}^{k\mu} = i\omega\widetilde{T}^{0\mu}\ . \tag{6.81}$$

Thus,

$$\begin{aligned}
\int d^3y\ \widetilde{T}^{ij}(\omega,\mathbf{y}) &= i\omega\int y^i\widetilde{T}^{0j}\ d^3y \\
&= \frac{i\omega}{2}\int (y^i\widetilde{T}^{0j} + y^j\widetilde{T}^{0i})\ d^3y \\
&= \frac{i\omega}{2}\int \left[\partial_l(y^iy^j\widetilde{T}^{0l}) - y^iy^j(\partial_l\widetilde{T}^{0l})\right]\ d^3y \\
&= -\frac{\omega^2}{2}\int y^iy^j\widetilde{T}^{00}\ d^3y\ .
\end{aligned} \tag{6.82}$$

The second line is justified since we know that the left hand side is symmetric in i and j, while the third and fourth lines are simply repetitions of reverse integration by parts and conservation of $T^{\mu\nu}$. It is conventional to define the **quadrupole moment tensor** of the energy density of the source,

$$q_{ij}(t) = 3\int y^iy^jT^{00}(t,\mathbf{y})\ d^3y\ , \tag{6.83}$$

a constant tensor on each surface of constant time. In terms of the Fourier transform of the quadrupole moment, our solution takes on the compact form

$$\widetilde{\bar{h}}_{ij}(\omega,\mathbf{x}) = -\frac{2G\omega^2}{3}\frac{e^{i\omega R}}{R}\widetilde{q}_{ij}(\omega)\ , \tag{6.84}$$

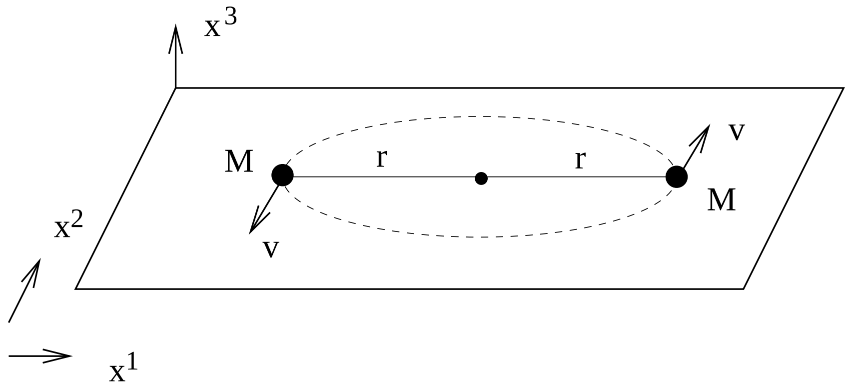
or, transforming back to t,

$$\begin{aligned}
\bar{h}_{ij}(t,\mathbf{x}) &= -\frac{1}{\sqrt{2\pi}}\frac{2G}{3R}\int d\omega\ e^{-i\omega(t-R)}\omega^2\widetilde{q}_{ij}(\omega) \\
&= \frac{1}{\sqrt{2\pi}}\frac{2G}{3R}\frac{d^2}{dt^2}\int d\omega\ e^{-i\omega t_r}\widetilde{q}_{ij}(\omega) \\
&= \frac{2G}{3R}\frac{d^2q_{ij}}{dt^2}(t_r)\ ,
\end{aligned} \tag{6.85}$$

where as before $t_r = t - R$.

The gravitational wave produced by an isolated nonrelativistic object is therefore proportional to the second derivative of the quadrupole moment of the energy density at the point where the past light cone of the observer intersects the source. In contrast, the leading contribution to electromagnetic radiation comes from the changing *dipole* moment of the charge density. The difference can be traced back to the universal nature of gravitation. A changing dipole moment corresponds to motion of the center of density — charge density in the case of electromagnetism, energy density in the case of gravitation. While there is nothing to stop the center of charge of an object from oscillating, oscillation of the center of mass of an isolated system violates conservation of momentum. (You can shake a body up and down, but you and the earth shake ever so slightly in the opposite direction to compensate.) The quadrupole moment, which measures the shape of the system, is generally smaller than the dipole moment, and for this reason (as well as the weak coupling of matter to gravity) gravitational radiation is typically much weaker than electromagnetic radiation.

It is always educational to take a general solution and apply it to a specific case of interest. One case of genuine interest is the gravitational radiation emitted by a binary star (two stars in orbit around each other). For simplicity let us consider two stars of mass M in a circular orbit in the x^1-x^2 plane, at distance r from their common center of mass.



We will treat the motion of the stars in the Newtonian approximation, where we can discuss their orbit just as Kepler would have. Circular orbits are most easily characterized by equating the force due to gravity to the outward "centrifugal" force:

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r} , \tag{6.86}$$

which gives us

$$v = \left(\frac{GM}{4r}\right)^{1/2} . \tag{6.87}$$

The time it takes to complete a single orbit is simply

$$T = \frac{2\pi r}{v} , \tag{6.88}$$

but more useful to us is the angular frequency of the orbit,

$$\Omega = \frac{2\pi}{T} = \left(\frac{GM}{4r^3}\right)^{1/2} . \tag{6.89}$$

In terms of Ω we can write down the explicit path of star a,

$$x_a^1 = r\cos\Omega t \ , \qquad x_a^2 = r\sin\Omega t \ , \tag{6.90}$$

and star b,

$$x_b^1 = -r\cos\Omega t \ , \qquad x_b^2 = -r\sin\Omega t \ . \tag{6.91}$$

The corresponding energy density is

$$T^{00}(t,\mathbf{x}) = M\delta(x^3)\left[\delta(x^1 - r\cos\Omega t)\delta(x^2 - r\sin\Omega t) + \delta(x^1 + r\cos\Omega t)\delta(x^2 + r\sin\Omega t)\right] . \tag{6.92}$$

The profusion of delta functions allows us to integrate this straightforwardly to obtain the quadrupole moment from (6.83):

$$\begin{aligned} q_{11} &= 6Mr^2\cos^2\Omega t = 3Mr^2(1+\cos 2\Omega t) \\ q_{22} &= 6Mr^2\sin^2\Omega t = 3Mr^2(1-\cos 2\Omega t) \\ q_{12} = q_{21} &= 6Mr^2(\cos\Omega t)(\sin\Omega t) = 3Mr^2\sin 2\Omega t \\ q_{i3} &= 0 \ . \end{aligned} \tag{6.93}$$

From this in turn it is easy to get the components of the metric perturbation from (6.85):

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{8GM}{R}\Omega^2 r^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{6.94}$$

The remaining components of $\bar{h}_{\mu\nu}$ could be derived from demanding that the harmonic gauge condition be satisfied. (We have not imposed a subsidiary gauge condition, so we are still free to do so.)

It is natural at this point to talk about the energy emitted via gravitational radiation. Such a discussion, however, is immediately beset by problems, both technical and philosophical. As we have mentioned before, there is no true local measure of the energy in the gravitational field. Of course, in the weak field limit, where we think of gravitation as being described by a symmetric tensor propagating on a fixed background metric, we might hope to derive an energy-momentum tensor for the fluctuations $h_{\mu\nu}$, just as we would for electromagnetism or any other field theory. To some extent this is possible, but there are still difficulties. As a result of these difficulties there are a number of different proposals in the literature for what we should use as the energy-momentum tensor for gravitation in the weak field limit; all of them are different, but for the most part they give the same answers for physically well-posed questions such as the rate of energy emitted by a binary system.

At a technical level, the difficulties begin to arise when we consider what form the energy-momentum tensor should take. We have previously mentioned the energy-momentum tensors for electromagnetism and scalar field theory, and they both shared an important feature — they were quadratic in the relevant fields. By hypothesis our approach to the weak field limit has been to only keep terms which are linear in the metric perturbation. Hence, in order to keep track of the energy carried by the gravitational waves, we will have to extend our calculations to at least second order in $h_{\mu\nu}$. In fact we have been cheating slightly all along. In discussing the effects of gravitational waves on test particles, and the generation of waves by a binary system, we have been using the fact that test particles move along geodesics. But as we know, this is derived from the covariant conservation of energy-momentum, $\nabla_\mu T^{\mu\nu} = 0$. In the order to which we have been working, however, we actually have $\partial_\mu T^{\mu\nu} = 0$, which would imply that test particles move on straight lines in the flat background metric. This is a symptom of the fundamental inconsistency of the weak field limit. In practice, the best that can be done is to solve the weak field equations to some appropriate order, and then justify after the fact the validity of the solution.

Keeping these issues in mind, let us consider Einstein's equations (in vacuum) to second order, and see how the result can be interpreted in terms of an energy-momentum tensor for the gravitational field. If we write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then at first order we have

$$G^{(1)}_{\mu\nu}[\eta + h] = 0 \ , \tag{6.95}$$

where $G^{(1)}_{\mu\nu}$ is Einstein's tensor expanded to first order in $h_{\mu\nu}$. These equations determine $h_{\mu\nu}$ up to (unavoidable) gauge transformations, so in order to satisfy the equations at second order we have to add a higher-order perturbation, and write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} \ . \tag{6.96}$$

The second-order version of Einstein's equations consists of all terms either quadratic in $h_{\mu\nu}$ or linear in $h^{(2)}_{\mu\nu}$. Since any cross terms would be of at least third order, we have

$$G^{(1)}_{\mu\nu}[\eta + h^{(2)}] + G^{(2)}_{\mu\nu}[\eta + h] = 0 \ . \tag{6.97}$$

Here, $G^{(2)}_{\mu\nu}$ is the part of the Einstein tensor which is of second order in the metric perturbation. It can be computed from the second-order Ricci tensor, which is given by

$$\begin{aligned} R^{(2)}_{\mu\nu} &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu)\partial_{[\sigma}h_{\rho]\mu} \\ &\quad + \frac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \frac{1}{4}(\partial_\rho h_{\mu\nu})\partial^\rho h - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho} \ . \end{aligned} \tag{6.98}$$

We can cast (6.97) into the suggestive form

$$G^{(1)}_{\mu\nu}[\eta + h^{(2)}] = 8\pi G t_{\mu\nu} \ , \tag{6.99}$$

simply by defining

$$t_{\mu\nu} = -\frac{1}{8\pi G} G^{(2)}_{\mu\nu}[\eta + h] \ . \tag{6.100}$$

The notation is of course meant to suggest that we think of $t_{\mu\nu}$ as an energy-momentum tensor, specifically that of the gravitational field (at least in the weak field regime). To make this claim seem plausible, note that the Bianchi identity for $G^{(1)}_{\mu\nu}[\eta + h^{(2)}]$ implies that $t_{\mu\nu}$ is conserved in the flat-space sense,

$$\partial_\mu t^{\mu\nu} = 0 \ . \tag{6.101}$$

Unfortunately there are some limitations on our interpretation of $t_{\mu\nu}$ as an energy-momentum tensor. Of course it is not a tensor at all in the full theory, but we are leaving that aside by hypothesis. More importantly, it is not invariant under gauge transformations (infinitesimal diffeomorphisms), as you could check by direct calculation. However, we can construct global quantities which are invariant under certain special kinds of gauge transformations (basically, those that vanish sufficiently rapidly at infinity; see Wald). These include the total energy on a surface Σ of constant time,

$$E = \int_\Sigma t_{00} \ d^3x \ , \tag{6.102}$$

and the total energy radiated through to infinity,

$$\Delta E = \int_S t_{0\mu} n^\mu \ d^2x \ dt \ . \tag{6.103}$$

Here, the integral is taken over a timelike surface S made of a spacelike two-sphere at infinity and some interval in time, and n^μ is a unit spacelike vector normal to S.

Evaluating these formulas in terms of the quadrupole moment of a radiating source involves a lengthy calculation which we will not reproduce here. Without further ado, the amount of radiated energy can be written

$$\Delta E = \int P \ dt \ , \tag{6.104}$$

where the power P is given by

$$P = \frac{G}{45} \left[\frac{d^3 Q^{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} \right]_{t_r} \ , \tag{6.105}$$

and here Q_{ij} is the traceless part of the quadrupole moment,

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} q_{kl} \ . \tag{6.106}$$

For the binary system represented by (6.93), the traceless part of the quadrupole is

$$Q_{ij} = Mr^2 \begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix} , \tag{6.107}$$

and its third time derivative is therefore

$$\frac{d^3 Q_{ij}}{dt^3} = 24Mr^2\Omega^3 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{6.108}$$

The power radiated by the binary is thus

$$P = \frac{2^7}{5} GM^2 r^4 \Omega^6 , \tag{6.109}$$

or, using expression (6.89) for the frequency,

$$P = \frac{2}{5}\frac{G^4 M^5}{r^5} . \tag{6.110}$$

Of course, this has actually been observed. In 1974 Hulse and Taylor discovered a binary system, PSR1913+16, in which both stars are very small (so classical effects are negligible, or at least under control) and one is a pulsar. The period of the orbit is eight hours, extremely small by astrophysical standards. The fact that one of the stars is a pulsar provides a very accurate clock, with respect to which the change in the period as the system loses energy can be measured. The result is consistent with the prediction of general relativity for energy loss through gravitational radiation. Hulse and Taylor were awarded the Nobel Prize in 1993 for their efforts.

7 The Schwarzschild Solution and Black Holes

We now move from the domain of the weak-field limit to solutions of the full nonlinear Einstein's equations. With the possible exception of Minkowski space, by far the most important such solution is that discovered by Schwarzschild, which describes spherically symmetric vacuum spacetimes. Since we are in vacuum, Einstein's equations become $R_{\mu\nu} = 0$. Of course, if we have a proposed solution to a set of differential equations such as this, it would suffice to plug in the proposed solution in order to verify it; we would like to do better, however. In fact, we will sketch a proof of Birkhoff's theorem, which states that the Schwarzschild solution is the *unique* spherically symmetric solution to Einstein's equations in vacuum. The procedure will be to first present some non-rigorous arguments that any spherically symmetric metric (whether or not it solves Einstein's equations) must take on a certain form, and then work from there to more carefully derive the actual solution in such a case.

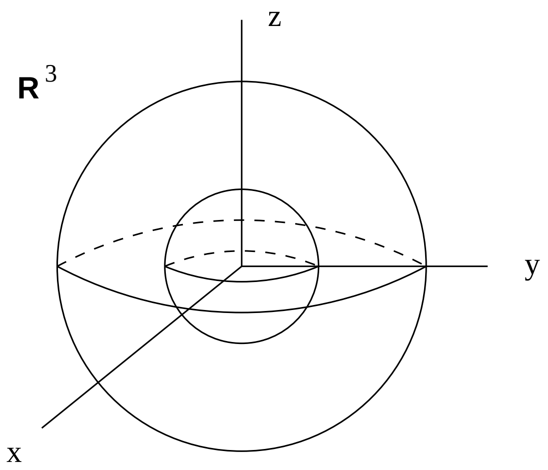
"Spherically symmetric" means "having the same symmetries as a sphere." (In this section the word "sphere" means S^2, not spheres of higher dimension.) Since the object of interest to us is the metric on a differentiable manifold, we are concerned with those metrics that have such symmetries. We know how to characterize symmetries of the metric — they are given by the existence of Killing vectors. Furthermore, we know what the Killing vectors of S^2 are, and that there are three of them. Therefore, a spherically symmetric manifold is one that has three Killing vector fields which are just like those on S^2. By "just like" we mean that the commutator of the Killing vectors is the same in either case — in fancier language, that the algebra generated by the vectors is the same. Something that we didn't show, but is true, is that we can choose our three Killing vectors on S^2 to be $(V^{(1)}, V^{(2)}, V^{(3)})$, such that

$$\begin{aligned} [V^{(1)}, V^{(2)}] &= V^{(3)} \\ [V^{(2)}, V^{(3)}] &= V^{(1)} \\ [V^{(3)}, V^{(1)}] &= V^{(2)} . \end{aligned} \tag{7.1}$$

The commutation relations are exactly those of SO(3), the group of rotations in three dimensions. This is no coincidence, of course, but we won't pursue this here. All we need is that a spherically symmetric manifold is one which possesses three Killing vector fields with the above commutation relations.

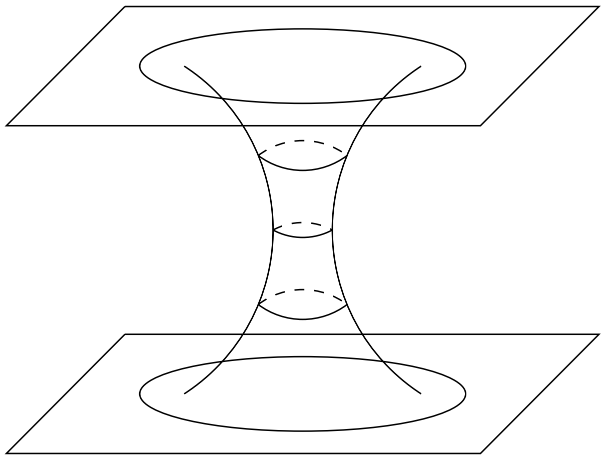
Back in section three we mentioned Frobenius's Theorem, which states that if you have a set of commuting vector fields then there exists a set of coordinate functions such that the vector fields are the partial derivatives with respect to these functions. In fact the theorem does not stop there, but goes on to say that if we have some vector fields which do *not* commute, but whose commutator closes — the commutator of any two fields in the set is a linear combination of other fields in the set — then the integral curves of these vector fields "fit together" to describe submanifolds of the manifold on which they are all defined. The dimensionality of the submanifold may be smaller than the number of vectors, or it could be equal, but obviously not larger. Vector fields which obey (7.1) will of course form 2-spheres. Since the vector fields stretch throughout the space, every point will be on exactly one of these spheres. (Actually, it's almost every point — we will show below how it can fail to be absolutely every point.) Thus, we say that a spherically symmetric manifold can be **foliated** into spheres.

Let's consider some examples to bring this down to earth. The simplest example is flat three-dimensional Euclidean space. If we pick an origin, then $\mathbf{R}^3$ is clearly spherically symmetric with respect to rotations around this origin. Under such rotations (*i.e.*, under the flow of the Killing vector fields) points move into each other, but each point stays on an S^2 at a fixed distance from the origin.



It is these spheres which foliate $\mathbf{R}^3$. Of course, they don't really foliate all of the space, since the origin itself just stays put under rotations — it doesn't move around on some two-sphere. But it should be clear that almost all of the space is properly foliated, and this will turn out to be enough for us.

We can also have spherical symmetry without an "origin" to rotate things around. An example is provided by a "wormhole", with topology $\mathbf{R} \times S^2$. If we suppress a dimension and draw our two-spheres as circles, such a space might look like this:



In this case the entire manifold can be foliated by two-spheres.

This foliated structure suggests that we put coordinates on our manifold in a way which is adapted to the foliation. By this we mean that, if we have an n-dimensional manifold foliated by m-dimensional submanifolds, we can use a set of m coordinate functions u^i on the submanifolds and a set of $n-m$ coordinate functions v^I to tell us which submanifold we are on. (So i runs from 1 to m, while I runs from 1 to $n-m$.) Then the collection of v's and u's coordinatize the entire space. If the submanifolds are maximally symmetric spaces (as two-spheres are), then there is the following powerful theorem: it is always possible to choose the u-coordinates such that the metric on the entire manifold is of the form

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{IJ}(v)\mathrm{d}v^I\mathrm{d}v^J + f(v)\gamma_{ij}(u)\mathrm{d}u^i\mathrm{d}u^j \ . \tag{7.2}$$

Here $\gamma_{ij}(u)$ is the metric on the submanifold. This theorem is saying two things at once: that there are no cross terms $\mathrm{d}v^I\mathrm{d}u^j$, and that both $g_{IJ}(v)$ and $f(v)$ are functions of the v^I alone, independent of the u^i. Proving the theorem is a mess, but you are encouraged to look in chapter 13 of Weinberg. Nevertheless, it is a perfectly sensible result. Roughly speaking, if g_{IJ} or f depended on the u^i then the metric would change as we moved in a single submanifold, which violates the assumption of symmetry. The unwanted cross terms, meanwhile, can be eliminated by making sure that the tangent vectors $\partial/\partial v^I$ are orthogonal to the submanifolds — in other words, that we line up our submanifolds in the same way throughout the space.

We are now through with handwaving, and can commence some honest calculation. For the case at hand, our submanifolds are two-spheres, on which we typically choose coordinates (θ, ϕ) in which the metric takes the form

$$d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \ \mathrm{d}\phi^2 \ . \tag{7.3}$$

Since we are interested in a four-dimensional spacetime, we need two more coordinates, which we can call a and b. The theorem (7.2) is then telling us that the metric on a spherically symmetric spacetime can be put in the form

$$ds^2 = g_{aa}(a,b)\mathrm{d}a^2 + g_{ab}(a,b)(\mathrm{d}a\mathrm{d}b + \mathrm{d}b\mathrm{d}a) + g_{bb}(a,b)\mathrm{d}b^2 + r^2(a,b)d\Omega^2 \ . \tag{7.4}$$

Here $r(a,b)$ is some as-yet-undetermined function, to which we have merely given a suggestive label. There is nothing to stop us, however, from changing coordinates from (a,b) to (a,r), by inverting $r(a,b)$. (The one thing that could possibly stop us would be if r were a function of a alone; in this case we could just as easily switch to (b,r), so we will not consider this situation separately.) The metric is then

$$ds^2 = g_{aa}(a,r)\mathrm{d}a^2 + g_{ar}(a,r)(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + g_{rr}(a,r)\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.5}$$

Our next step is to find a function $t(a,r)$ such that, in the (t,r) coordinate system, there are no cross terms $\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t$ in the metric. Notice that

$$\mathrm{d}t = \frac{\partial t}{\partial a}\mathrm{d}a + \frac{\partial t}{\partial r}\mathrm{d}r \ , \tag{7.6}$$

so

$$\mathrm{d}t^2 = \left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2 \ . \tag{7.7}$$

We would like to replace the first three terms in the metric (7.5) by

$$m\mathrm{d}t^2 + n\mathrm{d}r^2 \ , \tag{7.8}$$

for some functions m and n. This is equivalent to the requirements

$$m\left(\frac{\partial t}{\partial a}\right)^2 = g_{aa} \ , \tag{7.9}$$

$$n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr} \ , \tag{7.10}$$

and

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar} \ . \tag{7.11}$$

We therefore have three equations for the three unknowns $t(a,r)$, $m(a,r)$, and $n(a,r)$, just enough to determine them precisely (up to initial conditions for t). (Of course, they are "determined" in terms of the unknown functions g_{aa}, g_{ar}, and g_{rr}, so in this sense they are still undetermined.) We can therefore put our metric in the form

$$ds^2 = m(t,r)\mathrm{d}t^2 + n(t,r)\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.12}$$

To this point the only difference between the two coordinates t and r is that we have chosen r to be the one which multiplies the metric for the two-sphere. This choice was motivated by what we know about the metric for flat Minkowski space, which can be written $ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 d\Omega^2$. We know that the spacetime under consideration is Lorentzian, so either m or n will have to be negative. Let us choose m, the coefficient of $\mathrm{d}t^2$, to be negative. This is not a choice we are simply allowed to make, and in fact we will see later that it can go wrong, but we will assume it for now. The assumption is not completely unreasonable, since we know that Minkowski space is itself spherically symmetric, and will therefore be described by (7.12). With this choice we can trade in the functions m and n for new functions α and β, such that

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.13}$$

This is the best we can do for a general metric in a spherically symmetric spacetime. The next step is to actually solve Einstein's equations, which will allow us to determine explicitly the functions $\alpha(t,r)$ and $\beta(t,r)$. It is unfortunately necessary to compute the Christoffel symbols for (7.13), from which we can get the curvature tensor and thus the Ricci tensor. If we use labels $(0,1,2,3)$ for (t,r,θ,ϕ) in the usual way, the Christoffel symbols are given by

$$\begin{array}{ccc}
\Gamma^0_{00} = \partial_0\alpha & \Gamma^0_{01} = \partial_1\alpha & \Gamma^0_{11} = e^{2(\beta-\alpha)}\partial_0\beta \\
\Gamma^1_{00} = e^{2(\alpha-\beta)}\partial_1\alpha & \Gamma^1_{01} = \partial_0\beta & \Gamma^1_{11} = \partial_1\beta \\
\Gamma^2_{12} = \frac{1}{r} & \Gamma^1_{22} = -re^{-2\beta} & \Gamma^3_{13} = \frac{1}{r} \\
\Gamma^1_{33} = -re^{-2\beta}\sin^2\theta & \Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta} \ .
\end{array} \tag{7.14}$$

(Anything not written down explicitly is meant to be zero, or related to what is written by symmetries.) From these we get the following nonvanishing components of the Riemann tensor:

$$\begin{aligned}
R^0{}_{101} &= e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] + [\partial_1\alpha\partial_1\beta - \partial_1^2\alpha - (\partial_1\alpha)^2] \\
R^0{}_{202} &= -re^{-2\beta}\partial_1\alpha \\
R^0{}_{303} &= -re^{-2\beta}\sin^2\theta\ \partial_1\alpha \\
R^0{}_{212} &= -re^{-2\alpha}\partial_0\beta \\
R^0{}_{313} &= -re^{-2\alpha}\sin^2\theta\ \partial_0\beta \\
R^1{}_{212} &= re^{-2\beta}\partial_1\beta \\
R^1{}_{313} &= re^{-2\beta}\sin^2\theta\ \partial_1\beta \\
R^2{}_{323} &= (1 - e^{-2\beta})\sin^2\theta \ .
\end{aligned} \tag{7.15}$$

Taking the contraction as usual yields the Ricci tensor:

$$R_{00} = [\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] + e^{2(\alpha-\beta)}[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\partial_1\beta + \frac{2}{r}\partial_1\alpha]$$

$$
\begin{aligned}
R_{11} &= -[\partial_1^2\alpha + (\partial_1\alpha)^2 - \partial_1\alpha\partial_1\beta - \frac{2}{r}\partial_1\beta] + e^{2(\beta-\alpha)}[\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta] \\
R_{01} &= \frac{2}{r}\partial_0\beta \\
R_{22} &= e^{-2\beta}[r(\partial_1\beta - \partial_1\alpha) - 1] + 1 \\
R_{33} &= R_{22}\sin^2\theta \ .
\end{aligned} \tag{7.16}
$$

Our job is to set $R_{\mu\nu} = 0$. From $R_{01} = 0$ we get

$$
\partial_0\beta = 0 \ . \tag{7.17}
$$

If we consider taking the time derivative of $R_{22} = 0$ and using $\partial_0\beta = 0$, we get

$$
\partial_0\partial_1\alpha = 0 \ . \tag{7.18}
$$

We can therefore write

$$
\begin{aligned}
\beta &= \beta(r) \\
\alpha &= f(r) + g(t) \ .
\end{aligned} \tag{7.19}
$$

The first term in the metric (7.13) is therefore $-e^{2f(r)}e^{2g(t)}\mathrm{d}t^2$. But we could always simply redefine our time coordinate by replacing $\mathrm{d}t \to e^{-g(t)}\mathrm{d}t$; in other words, we are free to choose t such that $g(t) = 0$, whence $\alpha(t, r) = f(r)$. We therefore have

$$
ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{\beta(r)}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.20}
$$

All of the metric components are independent of the coordinate t. We have therefore proven a crucial result: *any spherically symmetric vacuum metric possesses a timelike Killing vector.*

This property is so interesting that it gets its own name: a metric which possesses a timelike Killing vector is called **stationary**. There is also a more restrictive property: a metric is called **static** if it possesses a timelike Killing vector which is orthogonal to a family of hypersurfaces. (A hypersurface in an n-dimensional manifold is simply an $(n-1)$-dimensional submanifold.) The metric (7.20) is not only stationary, but also static; the Killing vector field ∂_0 is orthogonal to the surfaces $t = const$ (since there are no cross terms such as $\mathrm{d}t\mathrm{d}r$ and so on). Roughly speaking, a static metric is one in which nothing is moving, while a stationary metric allows things to move but only in a symmetric way. For example, the static spherically symmetric metric (7.20) will describe non-rotating stars or black holes, while rotating systems (which keep rotating in the same way at all times) will be described by stationary metrics. It's hard to remember which word goes with which concept, but the distinction between the two concepts should be understandable.

Let's keep going with finding the solution. Since both R_{00} and R_{11} vanish, we can write

$$
0 = e^{2(\beta-\alpha)}R_{00} + R_{11} = \frac{2}{r}(\partial_1\alpha + \partial_1\beta) \ , \tag{7.21}
$$

which implies $\alpha = -\beta +$ constant. Once again, we can get rid of the constant by scaling our coordinates, so we have

$$\alpha = -\beta \ . \tag{7.22}$$

Next let us turn to $R_{22} = 0$, which now reads

$$e^{2\alpha}(2r\partial_1\alpha + 1) = 1 \ . \tag{7.23}$$

This is completely equivalent to

$$\partial_1(re^{2\alpha}) = 1 \ . \tag{7.24}$$

We can solve this to obtain

$$e^{2\alpha} = 1 + \frac{\mu}{r} \ , \tag{7.25}$$

where μ is some undetermined constant. With (7.22) and (7.25), our metric becomes

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) \mathrm{d}t^2 + \left(1 + \frac{\mu}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.26}$$

We now have no freedom left except for the single constant μ, so this form better solve the remaining equations $R_{00} = 0$ and $R_{11} = 0$; it is straightforward to check that it does, for any value of μ.

The only thing left to do is to interpret the constant μ in terms of some physical parameter. The most important use of a spherically symmetric vacuum solution is to represent the spacetime outside a star or planet or whatnot. In that case we would expect to recover the weak field limit as $r \to \infty$. In this limit, (7.26) implies

$$\begin{aligned} g_{00}(r \to \infty) &= -\left(1 + \frac{\mu}{r}\right) \ , \\ g_{rr}(r \to \infty) &= \left(1 - \frac{\mu}{r}\right) \ . \end{aligned} \tag{7.27}$$

The weak field limit, on the other hand, has

$$\begin{aligned} g_{00} &= -(1 + 2\Phi) \ , \\ g_{rr} &= (1 - 2\Phi) \ , \end{aligned} \tag{7.28}$$

with the potential $\Phi = -GM/r$. Therefore the metrics do agree in this limit, if we set $\mu = -2GM$.

Our final result is the celebrated **Schwarzschild metric**,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.29}$$

This is true for any spherically symmetric vacuum solution to Einstein's equations; M functions as a parameter, which we happen to know can be interpreted as the conventional

Newtonian mass that we would measure by studying orbits at large distances from the gravitating source. Note that as $M \to 0$ we recover Minkowski space, which is to be expected. Note also that the metric becomes progressively Minkowskian as we go to $r \to \infty$; this property is known as **asymptotic flatness**.

The fact that the Schwarzschild metric is not just a good solution, but is the unique spherically symmetric vacuum solution, is known as **Birkhoff's theorem**. It is interesting to note that the result is a static metric. We did not say anything about the source except that it be spherically symmetric. Specifically, we did not demand that the source itself be static; it could be a collapsing star, as long as the collapse were symmetric. Therefore a process such as a supernova explosion, which is basically spherical, would be expected to generate very little gravitational radiation (in comparison to the amount of energy released through other channels). This is the same result we would have obtained in electromagnetism, where the electromagnetic fields around a spherical charge distribution do not depend on the radial distribution of the charges.

Before exploring the behavior of test particles in the Schwarzschild geometry, we should say something about singularities. From the form of (7.29), the metric coefficients become infinite at $r = 0$ and $r = 2GM$ — an apparent sign that something is going wrong. The metric coefficients, of course, are coordinate-dependent quantities, and as such we should not make too much of their values; it is certainly possible to have a "coordinate singularity" which results from a breakdown of a specific coordinate system rather than the underlying manifold. An example occurs at the origin of polar coordinates in the plane, where the metric $ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ becomes degenerate and the component $g^{\theta\theta} = r^{-2}$ of the inverse metric blows up, even though that point of the manifold is no different from any other.

What kind of coordinate-independent signal should we look for as a warning that something about the geometry is out of control? This turns out to be a difficult question to answer, and entire books have been written about the nature of singularities in general relativity. We won't go into this issue in detail, but rather turn to one simple criterion for when something has gone wrong — when the curvature becomes infinite. The curvature is measured by the Riemann tensor, and it is hard to say when a tensor becomes infinite, since its components are coordinate-dependent. But from the curvature we can construct various scalar quantities, and since scalars are coordinate-independent it will be meaningful to say that they become infinite. This simplest such scalar is the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, but we can also construct higher-order scalars such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu}$, and so on. If any of these scalars (not necessarily all of them) go to infinity as we approach some point, we will regard that point as a singularity of the curvature. We should also check that the point is not "infinitely far away"; that is, that it can be reached by travelling a finite distance along a curve.

We therefore have a sufficient condition for a point to be considered a singularity. It is not a necessary condition, however, and it is generally harder to show that a given point is nonsingular; for our purposes we will simply test to see if geodesics are well-behaved at the point in question, and if so then we will consider the point nonsingular. In the case of the Schwarzschild metric (7.29), direct calculation reveals that

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12G^2M^2}{r^6} \ . \tag{7.30}$$

This is enough to convince us that $r = 0$ represents an honest singularity. At the other trouble spot, $r = 2GM$, you could check and see that none of the curvature invariants blows up. We therefore begin to think that it is actually not singular, and we have simply chosen a bad coordinate system. The best thing to do is to transform to more appropriate coordinates if possible. We will soon see that in this case it is in fact possible, and the surface $r = 2GM$ is very well-behaved (although interesting) in the Schwarzschild metric.

Having worried a little about singularities, we should point out that the behavior of Schwarzschild at $r \leq 2GM$ is of little day-to-day consequence. The solution we derived is valid only in vacuum, and we expect it to hold outside a spherical body such as a star. However, in the case of the Sun we are dealing with a body which extends to a radius of

$$R_\odot = 10^6 GM_\odot \ . \tag{7.31}$$

Thus, $r = 2GM_\odot$ is far inside the solar interior, where we do not expect the Schwarzschild metric to imply. In fact, realistic stellar interior solutions are of the form

$$ds^2 = -\left(1 - \frac{2Gm(r)}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.32}$$

See Schutz for details. Here $m(r)$ is a function of r which goes to zero faster than r itself, so there are no singularities to deal with at all. Nevertheless, there are objects for which the full Schwarzschild metric is required — black holes — and therefore we will let our imaginations roam far outside the solar system in this section.

The first step we will take to understand this metric more fully is to consider the behavior of geodesics. We need the nonzero Christoffel symbols for Schwarzschild:

$$\begin{array}{ccc} \Gamma^1_{00} = \frac{GM}{r^3}(r-2GM) & \Gamma^1_{11} = \frac{-GM}{r(r-2GM)} & \Gamma^0_{01} = \frac{GM}{r(r-2GM)} \\ \Gamma^2_{12} = \frac{1}{r} & \Gamma^1_{22} = -(r-2GM) & \Gamma^3_{13} = \frac{1}{r} \\ \Gamma^1_{33} = -(r-2GM)\sin^2\theta & \Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta} \ . \end{array} \tag{7.33}$$

The geodesic equation therefore turns into the following four equations, where λ is an affine parameter:

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \ , \tag{7.34}$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2$$
$$-(r-2GM)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0 \ , \tag{7.35}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \ , \tag{7.36}$$

and

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0 \ . \tag{7.37}$$

There does not seem to be much hope for simply solving this set of coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle; if K^μ is a Killing vector, we know that

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant} \ . \tag{7.38}$$

In addition, there is another constant of the motion that we always have for geodesics; metric compatibility implies that along the path the quantity

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{7.39}$$

is constant. Of course, for a massive particle we typically choose $\lambda = \tau$, and this relation simply becomes $\epsilon = -g_{\mu\nu}U^\mu U^\nu = +1$. For a massless particle we always have $\epsilon = 0$. We will also be concerned with spacelike geodesics (even though they do not correspond to paths of particles), for which we will choose $\epsilon = -1$.

Rather than immediately writing out explicit expressions for the four conserved quantities associated with Killing vectors, let's think about what they are telling us. Notice that the symmetries they represent are also present in flat spacetime, where the conserved quantities they lead to are very familiar. Invariance under time translations leads to conservation of energy, while invariance under spatial rotations leads to conservation of the three components of angular momentum. Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the direction of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it is. Thus, the two Killing vectors which lead to conservation of the direction of angular momentum imply

$$\theta = \frac{\pi}{2} \ . \tag{7.40}$$

The two remaining Killing vectors correspond to energy and the magnitude of angular momentum. The energy arises from the timelike Killing vector $K = \partial_t$, or

$$K_\mu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right) . \tag{7.41}$$

The Killing vector whose conserved quantity is the magnitude of the angular momentum is $L = \partial_\phi$, or

$$L_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right) . \tag{7.42}$$

Since (7.40) implies that $\sin\theta = 1$ along the geodesics of interest to us, the two conserved quantities are

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} = E , \tag{7.43}$$

and

$$r^2\frac{d\phi}{d\lambda} = L . \tag{7.44}$$

For massless particles these can be thought of as the energy and angular momentum; for massive particles they are the energy and angular momentum per unit mass of the particle. Note that the constancy of (7.44) is the GR equivalent of Kepler's second law (equal areas are swept out in equal times).

Together these conserved quantities provide a convenient way to understand the orbits of particles in the Schwarzschild geometry. Let us expand the expression (7.39) for ϵ to obtain

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon . \tag{7.45}$$

If we multiply this by $(1 - 2GM/r)$ and use our expressions for E and L, we obtain

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0 . \tag{7.46}$$

This is certainly progress, since we have taken a messy system of coupled equations and obtained a single equation for $r(\lambda)$. It looks even nicer if we rewrite it as

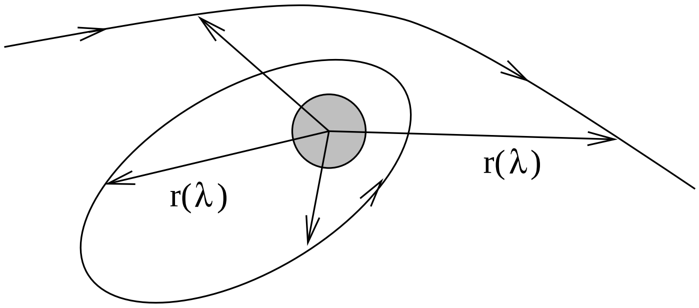
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 , \tag{7.47}$$

where

$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} . \tag{7.48}$$

In (7.47) we have precisely the equation for a classical particle of unit mass and "energy" $\frac{1}{2}E^2$ moving in a one-dimensional potential given by $V(r)$. (The true energy per unit mass is E, but the effective potential for the coordinate r responds to $\frac{1}{2}E^2$.)

Of course, our physical situation is quite different from a classical particle moving in one dimension. The trajectories under consideration are orbits around a star or other object:



The quantities of interest to us are not only $r(\lambda)$, but also $t(\lambda)$ and $\phi(\lambda)$. Nevertheless, we can go a long way toward understanding all of the orbits by understanding their radial behavior, and it is a great help to reduce this behavior to a problem we know how to solve.

A similar analysis of orbits in Newtonian gravity would have produced a similar result; the general equation (7.47) would have been the same, but the effective potential (7.48) would not have had the last term. (Note that this equation is not a power series in $1/r$, it is exact.) In the potential (7.48) the first term is just a constant, the second term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum which takes the same form in Newtonian gravity and general relativity. The last term, the GR contribution, will turn out to make a great deal of difference, especially at small r.

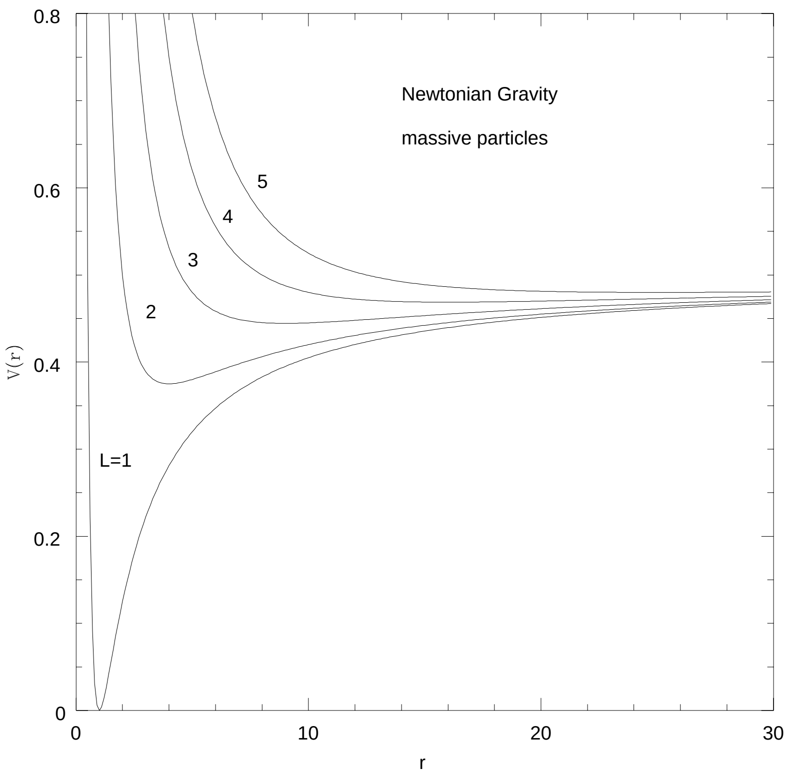
Let us examine the kinds of possible orbits, as illustrated in the figures. There are different curves $V(r)$ for different values of L; for any one of these curves, the behavior of the orbit can be judged by comparing the $\frac{1}{2}E^2$ to $V(r)$. The general behavior of the particle will be to move in the potential until it reaches a "turning point" where $V(r) = \frac{1}{2}E^2$, where it will begin moving in the other direction. Sometimes there may be no turning point to hit, in which case the particle just keeps going. In other cases the particle may simply move in a circular orbit at radius $r_c = const$; this can happen if the potential is flat, $dV/dr = 0$. Differentiating (7.48), we find that the circular orbits occur when

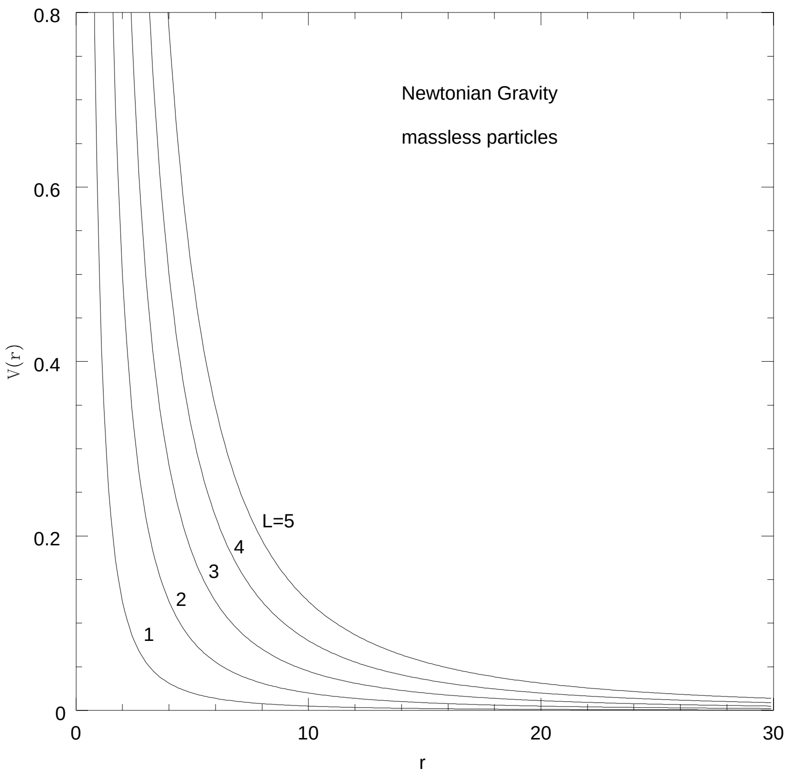
$$\epsilon GM r_c^2 - L^2 r_c + 3GML^2\gamma = 0 \ , \tag{7.49}$$

where $\gamma = 0$ in Newtonian gravity and $\gamma = 1$ in general relativity. Circular orbits will be stable if they correspond to a minimum of the potential, and unstable if they correspond to a maximum. Bound orbits which are not circular will oscillate around the radius of the stable circular orbit.

Turning to Newtonian gravity, we find that circular orbits appear at

$$r_c = \frac{L^2}{\epsilon GM} \ . \tag{7.50}$$


Newtonian Gravity
massive particles
5
4
3
2
L=1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r


Newtonian Gravity
massless particles
L=5
4
3
2
1
V(r)
0.8
0.6
0.4
0.2
0
0
10
20
30
r

For massless particles $\epsilon = 0$, and there are no circular orbits; this is consistent with the figure, which illustrates that there are no bound orbits of any sort. Although it is somewhat obscured in this coordinate system, massless particles actually move in a straight line, since the Newtonian gravitational force on a massless particle is zero. (Of course the standing of massless particles in Newtonian theory is somewhat problematic, but we will ignore that for now.) In terms of the effective potential, a photon with a given energy E will come in from $r = \infty$ and gradually "slow down" (actually $dr/d\lambda$ will decrease, but the speed of light isn't changing) until it reaches the turning point, when it will start moving away back to $r = \infty$. The lower values of L, for which the photon will come closer before it starts moving away, are simply those trajectories which are initially aimed closer to the gravitating body. For massive particles there will be stable circular orbits at the radius (7.50), as well as bound orbits which oscillate around this radius. If the energy is greater than the asymptotic value $E = 1$, the orbits will be unbound, describing a particle that approaches the star and then recedes. We know that the orbits in Newton's theory are conic sections — bound orbits are either circles or ellipses, while unbound ones are either parabolas or hyperbolas — although we won't show that here.

In general relativity the situation is different, but only for r sufficiently small. Since the difference resides in the term $-GML^2/r^3$, as $r \to \infty$ the behaviors are identical in the two theories. But as $r \to 0$ the potential goes to $-\infty$ rather than $+\infty$ as in the Newtonian case. At $r = 2GM$ the potential is always zero; inside this radius is the black hole, which we will discuss more thoroughly later. For massless particles there is always a barrier (except for $L = 0$, for which the potential vanishes identically), but a sufficiently energetic photon will nevertheless go over the barrier and be dragged inexorably down to the center. (Note that "sufficiently energetic" means "in comparison to its angular momentum" — in fact the frequency of the photon is immaterial, only the direction in which it is pointing.) At the top of the barrier there are unstable circular orbits. For $\epsilon = 0$, $\gamma = 1$, we can easily solve (7.49) to obtain

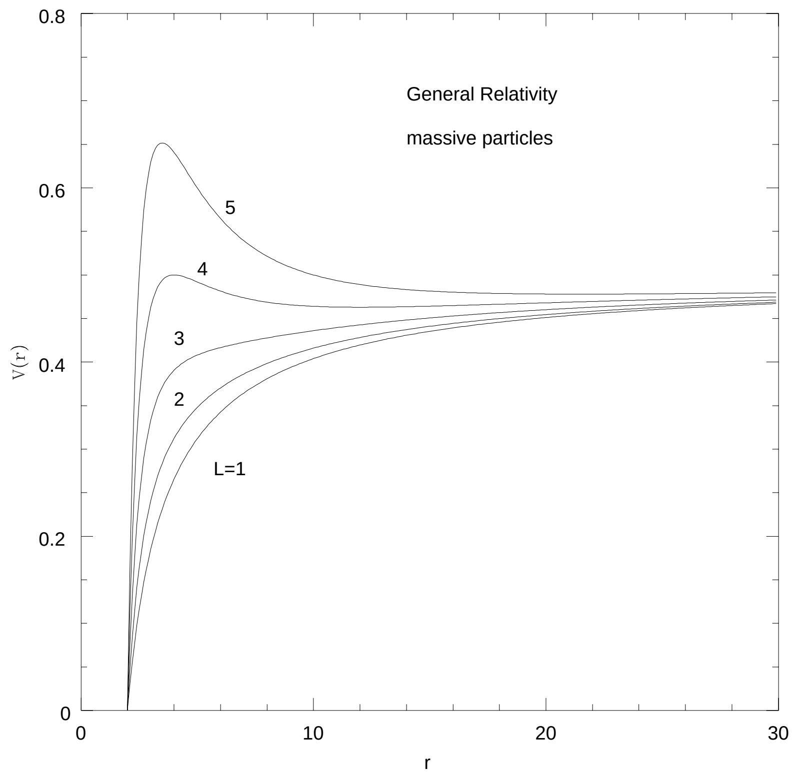
$$r_c = 3GM \ . \tag{7.51}$$

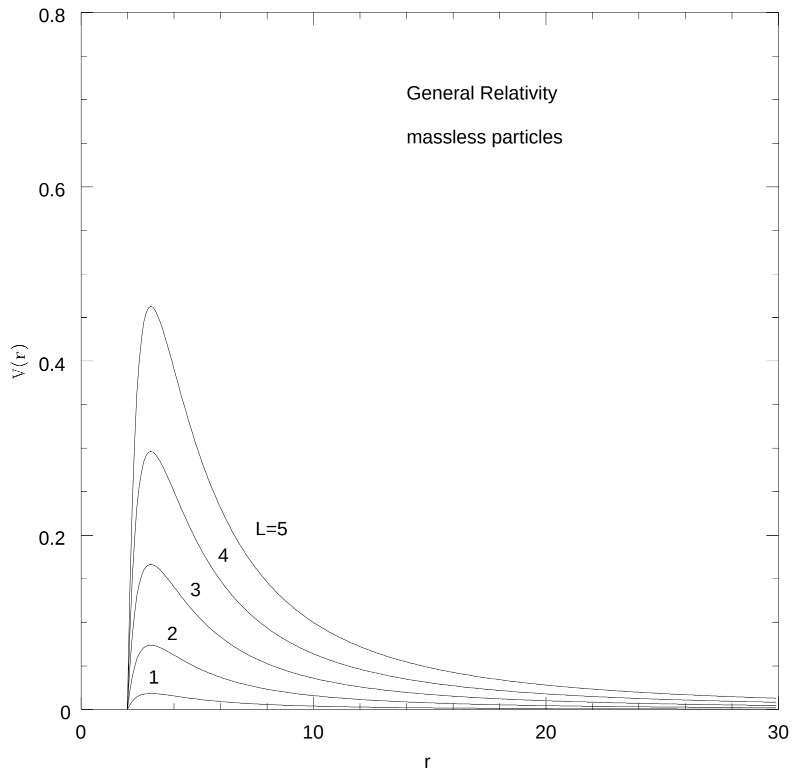
This is borne out by the figure, which shows a maximum of $V(r)$ at $r = 3GM$ for every L. This means that a photon can orbit forever in a circle at this radius, but any perturbation will cause it to fly away either to $r = 0$ or $r = \infty$.

For massive particles there are once again different regimes depending on the angular momentum. The circular orbits are at

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM} \ . \tag{7.52}$$

For large L there will be two circular orbits, one stable and one unstable. In the $L \to \infty$


General Relativity
massive particles
V(r)
r
L=1
2
3
4
5
0
0.2
0.4
0.6
0.8
0
10
20
30


General Relativity
massless particles
V(r)
r
L=5
4
3
2
1
0
0.2
0.4
0.6
0.8
0
10
20
30

limit their radii are given by

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM},\ 3GM\right) . \tag{7.53}$$

In this limit the stable circular orbit becomes farther and farther away, while the unstable one approaches $3GM$, behavior which parallels the massless case. As we decrease L the two circular orbits come closer together; they coincide when the discriminant in (7.52) vanishes, at

$$L = \sqrt{12}GM , \tag{7.54}$$

for which

$$r_c = 6GM , \tag{7.55}$$

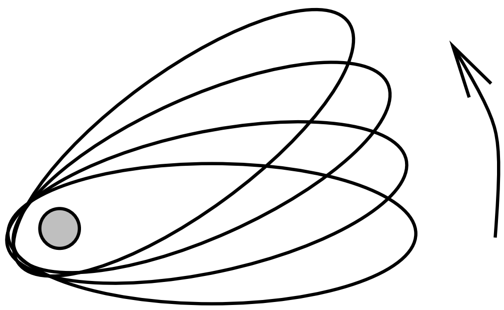
and disappear entirely for smaller L. Thus $6GM$ is the smallest possible radius of a stable circular orbit in the Schwarzschild metric. There are also unbound orbits, which come in from infinity and turn around, and bound but noncircular ones, which oscillate around the stable circular radius. Note that such orbits, which would describe exact conic sections in Newtonian gravity, will not do so in GR, although we would have to solve the equation for $d\phi/dt$ to demonstrate it. Finally, there are orbits which come in from infinity and continue all the way in to $r = 0$; this can happen either if the energy is higher than the barrier, or for $L < \sqrt{12}GM$, when the barrier goes away entirely.

We have therefore found that the Schwarzschild solution possesses stable circular orbits for $r > 6GM$ and unstable circular orbits for $3GM < r < 6GM$. It's important to remember that these are only the geodesics; there is nothing to stop an accelerating particle from dipping below $r = 3GM$ and emerging, as long as it stays beyond $r = 2GM$.

Most experimental tests of general relativity involve the motion of test particles in the solar system, and hence geodesics of the Schwarzschild metric; this is therefore a good place to pause and consider these tests. Einstein suggested three tests: the deflection of light, the precession of perihelia, and gravitational redshift. The deflection of light is observable in the weak-field limit, and therefore is not really a good test of the exact form of the Schwarzschild geometry. Observations of this deflection have been performed during eclipses of the Sun, with results which agree with the GR prediction (although it's not an especially clean experiment). The precession of perihelia reflects the fact that noncircular orbits are not closed ellipses; to a good approximation they are ellipses which precess, describing a flower pattern.

Using our geodesic equations, we could solve for $d\phi/d\lambda$ as a power series in the eccentricity e of the orbit, and from that obtain the apsidal frequency ω_a, defined as 2π divided by the time it takes for the ellipse to precess once around. For details you can look in Weinberg; the answer is

$$\omega_a = \frac{3(GM)^{3/2}}{c^2(1-e^2)r^{5/2}} , \tag{7.56}$$



where we have restored the c to make it easier to compare with observation. (It is a good exercise to derive this yourself to lowest nonvanishing order, in which case the e^2 is missing.) Historically the precession of Mercury was the first test of GR. For Mercury the relevant numbers are

$$\begin{aligned} \frac{GM_\odot}{c^2} &= 1.48 \times 10^5 \text{ cm} , \\ a &= 5.55 \times 10^{12} \text{ cm} , \end{aligned} \tag{7.57}$$

and of course $c = 3.00 \times 10^{10}$ cm/sec. This gives $\omega_a = 2.35 \times 10^{-14}$ sec^{-1}. In other words, the major axis of Mercury's orbit precesses at a rate of 42.9 arcsecs every 100 years. The observed value is 5601 arcsecs/100 yrs. However, much of that is due to the precession of equinoxes in our geocentric coordinate system; 5025 arcsecs/100 yrs, to be precise. The gravitational perturbations of the other planets contribute an additional 532 arcsecs/100 yrs, leaving 43 arcsecs/100 yrs to be explained by GR, which it does quite well.

The gravitational redshift, as we have seen, is another effect which is present in the weak field limit, and in fact will be predicted by any theory of gravity which obeys the Principle of Equivalence. However, this only applies to small enough regions of spacetime; over larger distances, the exact amount of redshift will depend on the metric, and thus on the theory under question. It is therefore worth computing the redshift in the Schwarzschild geometry. We consider two observers who are not moving on geodesics, but are stuck at fixed spatial coordinate values (r_1, θ_1, ϕ_1) and (r_2, θ_2, ϕ_2). According to (7.45), the proper time of observer i will be related to the coordinate time t by

$$\frac{d\tau_i}{dt} = \left(1 - \frac{2GM}{r_i}\right)^{1/2} . \tag{7.58}$$

Suppose that the observer $\mathcal{O}_1$ emits a light pulse which travels to the observer $\mathcal{O}_2$, such that $\mathcal{O}_1$ measures the time between two successive crests of the light wave to be $\Delta\tau_1$. Each crest follows the same path to $\mathcal{O}_2$, except that they are separated by a coordinate time

$$\Delta t = \left(1 - \frac{2GM}{r_1}\right)^{-1/2} \Delta\tau_1 . \tag{7.59}$$

This separation in coordinate time does not change along the photon trajectories, but the second observer measures a time between successive crests given by

$$\begin{aligned}\Delta\tau_2 &= \left(1-\frac{2GM}{r_2}\right)^{1/2}\Delta t \\ &= \left(\frac{1-2GM/r_2}{1-2GM/r_1}\right)^{1/2}\Delta\tau_1 \ .\end{aligned} \tag{7.60}$$

Since these intervals $\Delta\tau_i$ measure the proper time between two crests of an electromagnetic wave, the observed frequencies will be related by

$$\begin{aligned}\frac{\omega_2}{\omega_1} &= \frac{\Delta\tau_1}{\Delta\tau_2} \\ &= \left(\frac{1-2GM/r_1}{1-2GM/r_2}\right)^{1/2} \ .\end{aligned} \tag{7.61}$$

This is an exact result for the frequency shift; in the limit $r >> 2GM$ we have

$$\begin{aligned}\frac{\omega_2}{\omega_1} &= 1-\frac{GM}{r_1}+\frac{GM}{r_2} \\ &= 1+\Phi_1-\Phi_2 \ .\end{aligned} \tag{7.62}$$

This tells us that the frequency goes down as Φ increases, which happens as we climb out of a gravitational field; thus, a redshift. You can check that it agrees with our previous calculation based on the equivalence principle.

Since Einstein's proposal of the three classic tests, further tests of GR have been proposed. The most famous is of course the binary pulsar, discussed in the previous section. Another is the gravitational time delay, discovered by (and observed by) Shapiro. This is just the fact that the time elapsed along two different trajectories between two events need not be the same. It has been measured by reflecting radar signals off of Venus and Mars, and once again is consistent with the GR prediction. One effect which has not yet been observed is the Lense-Thirring, or frame-dragging effect. There has been a long-term effort devoted to a proposed satellite, dubbed Gravity Probe B, which would involve extraordinarily precise gyroscopes whose precession could be measured and the contribution from GR sorted out. It has a ways to go before being launched, however, and the survival of such projects is always year-to-year.

We now know something about the behavior of geodesics outside the troublesome radius $r = 2GM$, which is the regime of interest for the solar system and most other astrophysical situations. We will next turn to the study of objects which are described by the Schwarzschild solution even at radii smaller than $2GM$ — black holes. (We'll use the term "black hole" for the moment, even though we haven't introduced a precise meaning for such an object.)

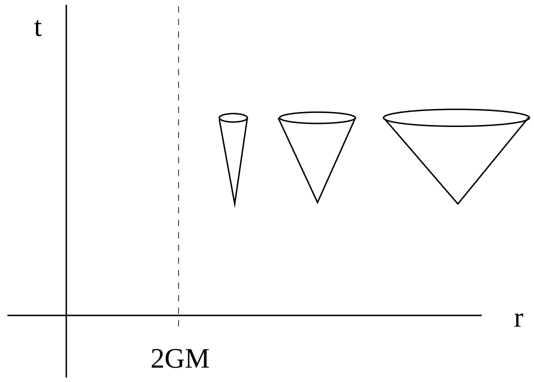
One way of understanding a geometry is to explore its causal structure, as defined by the light cones. We therefore consider radial null curves, those for which θ and ϕ are constant and $ds^2 = 0$:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \ , \tag{7.63}$$

from which we see that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1} \ . \tag{7.64}$$

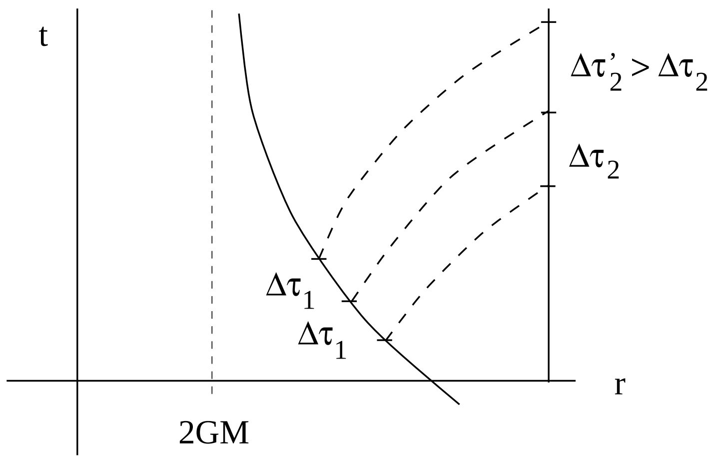
This of course measures the slope of the light cones on a spacetime diagram of the t-r plane. For large r the slope is ± 1, as it would be in flat space, while as we approach $r = 2GM$ we get $dt/dr \to \pm\infty$, and the light cones "close up":



Thus a light ray which approaches $r = 2GM$ never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius.

As we will see, this is an illusion, and the light ray (or a massive particle) actually has no trouble reaching $r = 2GM$. But an observer far away would never be able to tell. If we stayed outside while an intrepid observational general relativist dove into the black hole, sending back signals all the time, we would simply see the signals reach us more and more slowly. This should be clear from the pictures, and is confirmed by our computation of $\Delta\tau_1/\Delta\tau_2$ when we discussed the gravitational redshift (7.61). As infalling astronauts approach $r = 2GM$, any fixed interval $\Delta\tau_1$ of their proper time corresponds to a longer and longer interval $\Delta\tau_2$ from our point of view. This continues forever; we would never see the astronaut cross $r = 2GM$, we would just see them move more and more slowly (and become redder and redder, almost as if they were embarrassed to have done something as stupid as diving into a black hole).

The fact that we never see the infalling astronauts reach $r = 2GM$ is a meaningful statement, but the fact that their trajectory in the t-r plane never reaches there is not. It is highly dependent on our coordinate system, and we would like to ask a more coordinate-independent question (such as, do the astronauts reach this radius in a finite amount of their proper time?). The best way to do this is to change coordinates to a system which is better



behaved at $r = 2GM$. There does exist a set of such coordinates, which we now set out to find. There is no way to "derive" a coordinate transformation, of course, we just say what the new coordinates are and plug in the formulas. But we will develop these coordinates in several steps, in hopes of making the choices seem somewhat motivated.

The problem with our current coordinates is that $dt/dr \to \infty$ along radial null geodesics which approach $r = 2GM$; progress in the r direction becomes slower and slower with respect to the coordinate time t. We can try to fix this problem by replacing t with a coordinate which "moves more slowly" along null geodesics. First notice that we can explicitly solve the condition (7.64) characterizing radial null curves to obtain

$$t = \pm r^* + \text{ constant } , \tag{7.65}$$

where the **tortoise coordinate** r^* is defined by

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) . \tag{7.66}$$

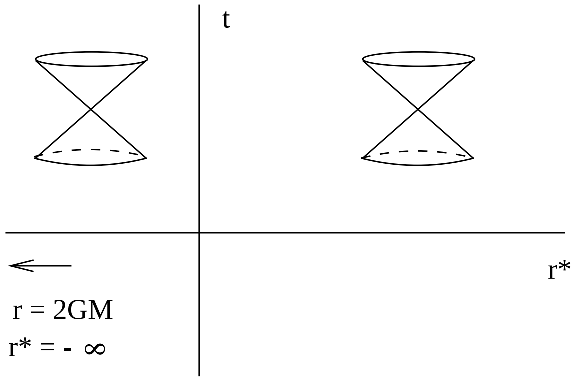
(The tortoise coordinate is only sensibly related to r when $r \geq 2GM$, but beyond there our coordinates aren't very good anyway.) In terms of the tortoise coordinate the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2 d\Omega^2 , \tag{7.67}$$

where r is thought of as a function of r^*. This represents some progress, since the light cones now don't seem to close up; furthermore, none of the metric coefficients becomes infinite at $r = 2GM$ (although both g_{tt} and $g_{r^*r^*}$ become zero). The price we pay, however, is that the surface of interest at $r = 2GM$ has just been pushed to infinity.

Our next move is to define coordinates which are naturally adapted to the null geodesics. If we let

$$\tilde{u} = t + r^*$$



$$\tilde{v} \;=\; t - r^* \;, \tag{7.68}$$

then infalling radial null geodesics are characterized by $\tilde{u}$ = constant, while the outgoing ones satisfy $\tilde{v}$ = constant. Now consider going back to the original radial coordinate r, but replacing the timelike coordinate t with the new coordinate $\tilde{u}$. These are known as **Eddington-Finkelstein coordinates**. In terms of them the metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}\tilde{u}^2 + (\mathrm{d}\tilde{u}\mathrm{d}r + \mathrm{d}r\mathrm{d}\tilde{u}) + r^2 d\Omega^2 \;. \tag{7.69}$$

Here we see our first sign of real progress. Even though the metric coefficient $g_{\tilde{u}\tilde{u}}$ vanishes at $r = 2GM$, there is no real degeneracy; the determinant of the metric is

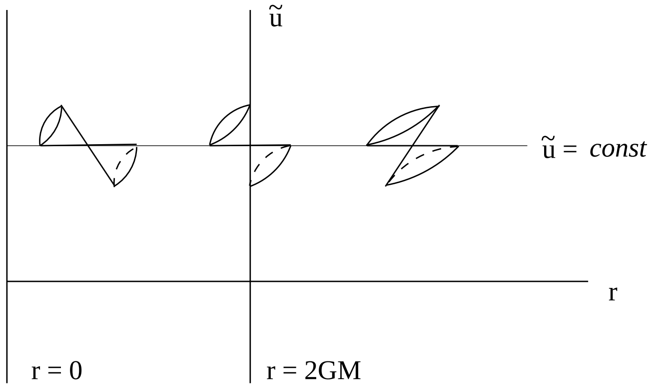
$$g = -r^4 \sin^2\theta \;, \tag{7.70}$$

which is perfectly regular at $r = 2GM$. Therefore the metric is invertible, and we see once and for all that $r = 2GM$ is simply a coordinate singularity in our original (t, r, θ, ϕ) system. In the Eddington-Finkelstein coordinates the condition for radial null curves is solved by

$$\frac{d\tilde{u}}{dr} = \begin{cases} 0 \;, & \text{(infalling)} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} \;. & \text{(outgoing)} \end{cases} \tag{7.71}$$

We can therefore see what has happened: in this coordinate system the light cones remain well-behaved at $r = 2GM$, and this surface is at a finite coordinate value. There is no problem in tracing the paths of null or timelike particles past the surface. On the other hand, something interesting is certainly going on. Although the light cones don't close up, they do tilt over, such that for $r < 2GM$ all future-directed paths are in the direction of decreasing r.

The surface $r = 2GM$, while being locally perfectly regular, globally functions as a point of no return — once a test particle dips below it, it can never come back. For this reason $r = 2GM$ is known as the **event horizon**; no event at $r \leq 2GM$ can influence any other



event at $r > 2GM$. Notice that the event horizon is a null surface, not a timelike one. Notice also that since nothing can escape the event horizon, it is impossible for us to "see inside" — thus the name **black hole**.

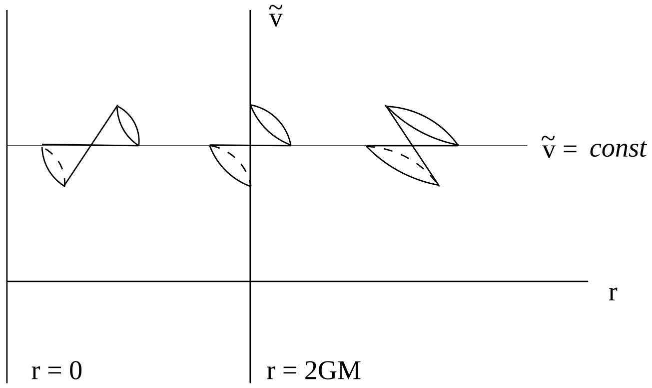
Let's consider what we have done. Acting under the suspicion that our coordinates may not have been good for the entire manifold, we have changed from our original coordinate t to the new one $\tilde{u}$, which has the nice property that if we decrease r along a radial curve null curve $\tilde{u} =$ constant, we go right through the event horizon without any problems. (Indeed, a local observer actually making the trip would not necessarily know when the event horizon had been crossed — the local geometry is no different than anywhere else.) We therefore conclude that our suspicion was correct and our initial coordinate system didn't do a good job of covering the entire manifold. The region $r \leq 2GM$ should certainly be included in our spacetime, since physical particles can easily reach there and pass through. However, there is no guarantee that we are finished; perhaps there are other directions in which we can extend our manifold.

In fact there are. Notice that in the $(\tilde{u}, r)$ coordinate system we can cross the event horizon on future-directed paths, but not on past-directed ones. This seems unreasonable, since we started with a time-independent solution. But we could have chosen $\tilde{v}$ instead of $\tilde{u}$, in which case the metric would have been

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}\tilde{v}^2 - (\mathrm{d}\tilde{v}\mathrm{d}r + \mathrm{d}r\mathrm{d}\tilde{v}) + r^2 d\Omega^2 \ . \tag{7.72}$$

Now we can once again pass through the event horizon, but this time only along past-directed curves.

This is perhaps a surprise: we can consistently follow either future-directed or past-directed curves through $r = 2GM$, but we arrive at different places. It was actually to be expected, since from the definitions (7.68), if we keep $\tilde{u}$ constant and decrease r we must have $t \to +\infty$, while if we keep $\tilde{v}$ constant and decrease r we must have $t \to -\infty$. (The tortoise coordinate r^* goes to $-\infty$ as $r \to 2GM$.) So we have extended spacetime in two different directions, one to the future and one to the past.



The next step would be to follow spacelike geodesics to see if we would uncover still more regions. The answer is yes, we would reach yet another piece of the spacetime, but let's shortcut the process by defining coordinates that are good all over. A first guess might be to use both $\tilde{u}$ and $\tilde{v}$ at once (in place of t and r), which leads to

$$ds^2 = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)(\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \mathrm{d}\tilde{v}\mathrm{d}\tilde{u}) + r^2 d\Omega^2 \ , \tag{7.73}$$

with r defined implicitly in terms of $\tilde{u}$ and $\tilde{v}$ by

$$\frac{1}{2}(\tilde{u} - \tilde{v}) = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \ . \tag{7.74}$$

We have actually re-introduced the degeneracy with which we started out; in these coordinates $r = 2GM$ is "infinitely far away" (at either $\tilde{u} = -\infty$ or $\tilde{v} = +\infty$). The thing to do is to change to coordinates which pull these points into finite coordinate values; a good choice is

$$\begin{aligned} u' &= e^{\tilde{u}/4GM} \\ v' &= e^{-\tilde{v}/4GM} \ , \end{aligned} \tag{7.75}$$

which in terms of our original (t, r) system is

$$\begin{aligned} u' &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM} \\ v' &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM} \ . \end{aligned} \tag{7.76}$$

In the (u', v', θ, ϕ) system the Schwarzschild metric is

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(\mathrm{d}u'\mathrm{d}v' + \mathrm{d}v'\mathrm{d}u') + r^2 d\Omega^2 \ . \tag{7.77}$$

Finally the nonsingular nature of $r = 2GM$ becomes completely manifest; in this form none of the metric coefficients behave in any special way at the event horizon.

Both u' and v' are null coordinates, in the sense that their partial derivatives $\partial/\partial u'$ and $\partial/\partial v'$ are null vectors. There is nothing wrong with this, since the collection of four partial derivative vectors (two null and two spacelike) in this system serve as a perfectly good basis for the tangent space. Nevertheless, we are somewhat more comfortable working in a system where one coordinate is timelike and the rest are spacelike. We therefore define

$$\begin{aligned} u &= \frac{1}{2}(u' - v') \\ &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh(t/4GM) \end{aligned} \tag{7.78}$$

and

$$\begin{aligned} v &= \frac{1}{2}(u' + v') \\ &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh(t/4GM) \ , \end{aligned} \tag{7.79}$$

in terms of which the metric becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dv^2 + du^2) + r^2 d\Omega^2 \ , \tag{7.80}$$

where r is defined implicitly from

$$(u^2 - v^2) = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \ . \tag{7.81}$$

The coordinates (v, u, θ, ϕ) are known as **Kruskal coordinates**, or sometimes Kruskal-Szekres coordinates. Note that v is the timelike coordinate.

The Kruskal coordinates have a number of miraculous properties. Like the (t, r^*) coordinates, the radial null curves look like they do in flat space:

$$v = \pm u + \text{constant} \ . \tag{7.82}$$

Unlike the (t, r^*) coordinates, however, the event horizon $r = 2GM$ is not infinitely far away; in fact it is defined by

$$v = \pm u \ , \tag{7.83}$$

consistent with it being a null surface. More generally, we can consider the surfaces $r =$ constant. From (7.81) these satisfy

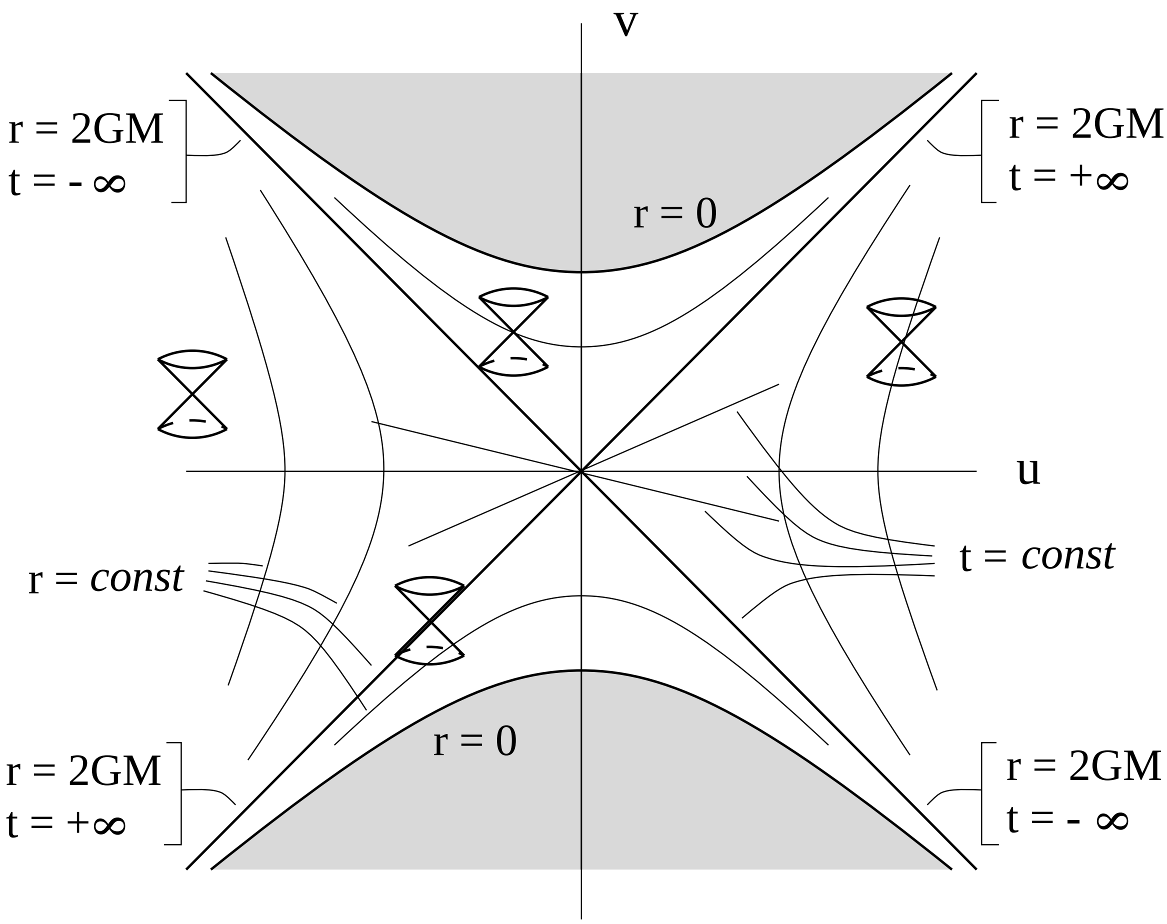
$$u^2 - v^2 = \text{ constant} \ . \tag{7.84}$$

Thus, they appear as hyperbolae in the u-v plane. Furthermore, the surfaces of constant t are given by

$$\frac{v}{u} = \tanh(t/4GM) \ , \tag{7.85}$$

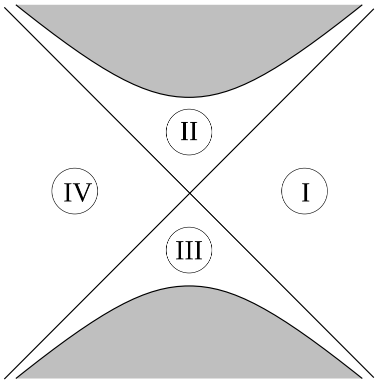
which defines straight lines through the origin with slope $\tanh(t/4GM)$. Note that as $t \to \pm\infty$ this becomes the same as (7.83); therefore these surfaces are the same as $r = 2GM$.

Now, our coordinates (v, u) should be allowed to range over every value they can take without hitting the real singularity at $r = 2GM$; the allowed region is therefore $-\infty \leq u \leq \infty$ and $v^2 < u^2 + 1$. We can now draw a spacetime diagram in the v-u plane (with θ and ϕ suppressed), known as a "Kruskal diagram", which represents the entire spacetime corresponding to the Schwarzschild metric.



Each point on the diagram is a two-sphere.

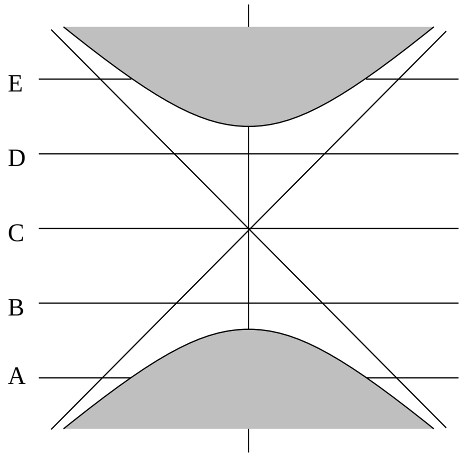
Our original coordinates (t, r) were only good for $r > 2GM$, which is only a part of the manifold portrayed on the Kruskal diagram. It is convenient to divide the diagram into four regions:



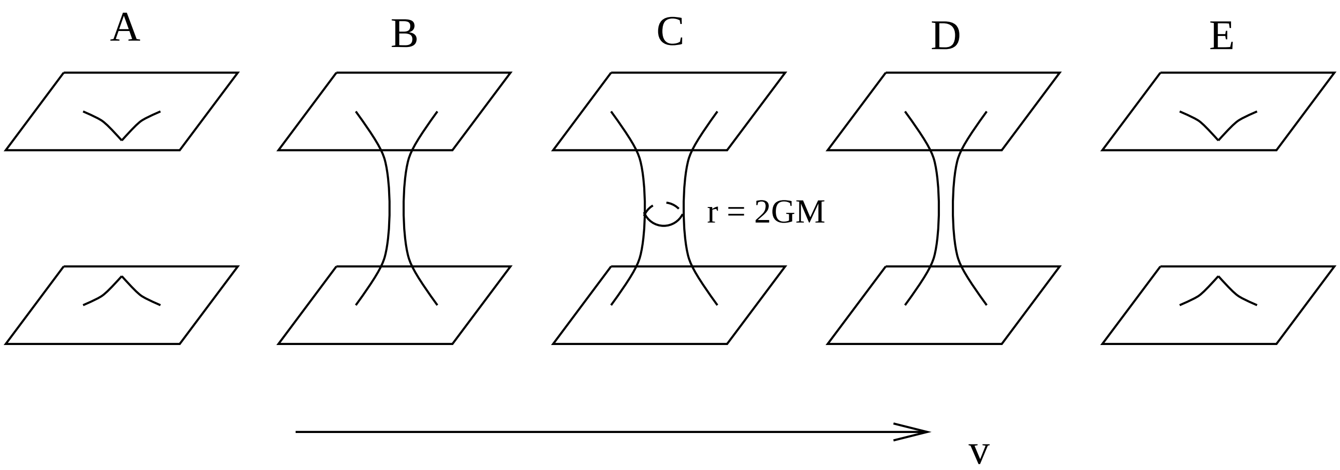
The region in which we started was region I; by following future-directed null rays we reached region II, and by following past-directed null rays we reached region III. If we had explored spacelike geodesics, we would have been led to region IV. The definitions (7.78) and (7.79) which relate (u, v) to (t, r) are really only good in region I; in the other regions it is necessary to introduce appropriate minus signs to prevent the coordinates from becoming imaginary.

Having extended the Schwarzschild geometry as far as it will go, we have described a remarkable spacetime. Region II, of course, is what we think of as the black hole. Once anything travels from region I into II, it can never return. In fact, every future-directed path in region II ends up hitting the singularity at $r = 0$; once you enter the event horizon, you are utterly doomed. This is worth stressing; not only can you not escape back to region I, you cannot even stop yourself from moving in the direction of decreasing r, since this is simply the timelike direction. (This could have been seen in our original coordinate system; for $r < 2GM$, t becomes spacelike and r becomes timelike.) Thus you can no more stop moving toward the singularity than you can stop getting older. Since proper time is maximized along a geodesic, you will live the longest if you don't struggle, but just relax as you approach the singularity. Not that you will have long to relax. (Nor that the voyage will be very relaxing; as you approach the singularity the tidal forces become infinite. As you fall toward the singularity your feet and head will be pulled apart from each other, while your torso is squeezed to infinitesimal thinness. The grisly demise of an astrophysicist falling into a black hole is detailed in Misner, Thorne, and Wheeler, section 32.6. Note that they use orthonormal frames [not that it makes the trip any more enjoyable].)

Regions III and IV might be somewhat unexpected. Region III is simply the time-reverse of region II, a part of spacetime from which things can escape to us, while we can never get there. It can be thought of as a "white hole." There is a singularity in the past, out of which the universe appears to spring. The boundary of region III is sometimes called the past event horizon, while the boundary of region II is called the future event horizon. Region IV, meanwhile, cannot be reached from our region I either forward or backward in time (nor can anybody from over there reach us). It is another asymptotically flat region of spacetime, a mirror image of ours. It can be thought of as being connected to region I by a "wormhole," a neck-like configuration joining two distinct regions. Consider slicing up the Kruskal diagram into spacelike surfaces of constant v:



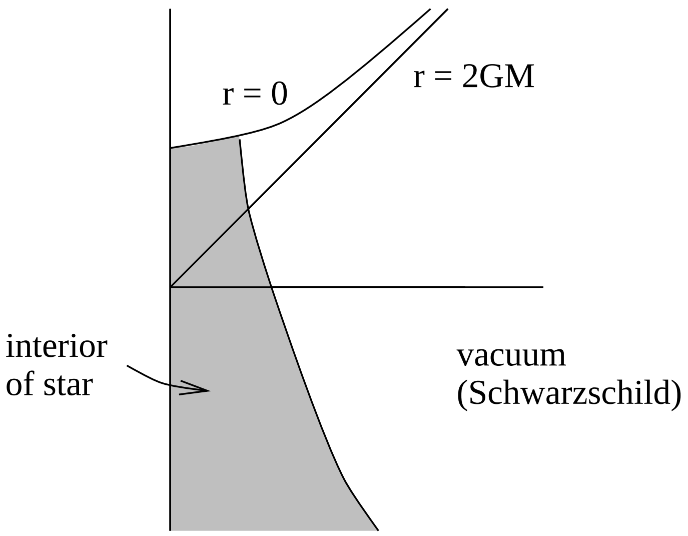
Now we can draw pictures of each slice, restoring one of the angular coordinates for clarity:



So the Schwarzschild geometry really describes two asymptotically flat regions which reach toward each other, join together via a wormhole for a while, and then disconnect. But the wormhole closes up too quickly for any timelike observer to cross it from one region into the next.

It might seem somewhat implausible, this story about two separate spacetimes reaching toward each other for a while and then letting go. In fact, it is not expected to happen in the real world, since the Schwarzschild metric does not accurately model the entire universe.

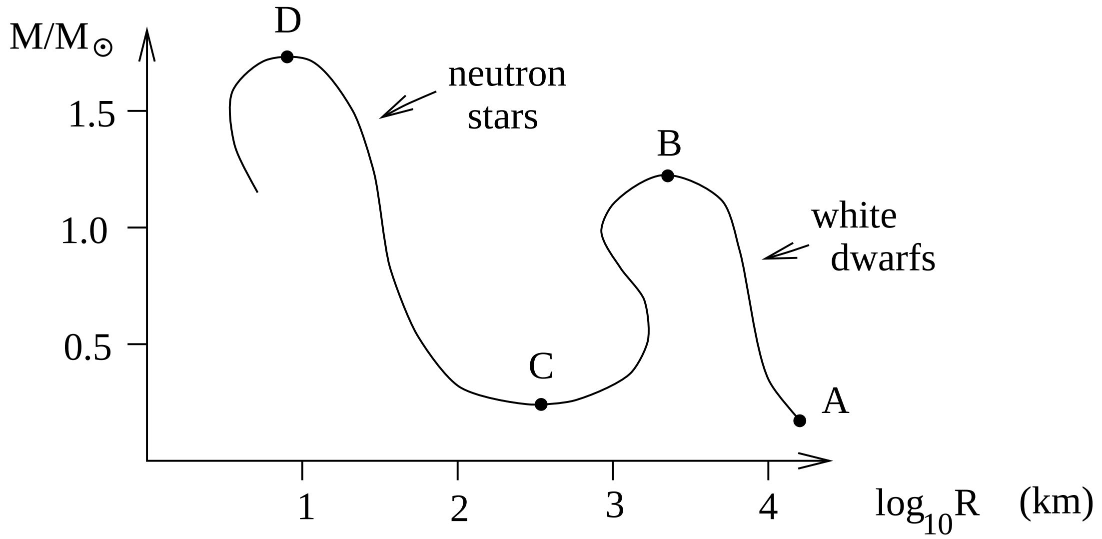
Remember that it is only valid in vacuum, for example outside a star. If the star has a radius larger than $2GM$, we need never worry about any event horizons at all. But we believe that there are stars which collapse under their own gravitational pull, shrinking down to below $r = 2GM$ and further into a singularity, resulting in a black hole. There is no need for a white hole, however, because the past of such a spacetime looks nothing like that of the full Schwarzschild solution. Roughly, a Kruskal-like diagram for stellar collapse would look like the following:



The shaded region is not described by Schwarzschild, so there is no need to fret about white holes and wormholes.

While we are on the subject, we can say something about the formation of astrophysical black holes from massive stars. The life of a star is a constant struggle between the inward pull of gravity and the outward push of pressure. When the star is burning nuclear fuel at its core, the pressure comes from the heat produced by this burning. (We should put "burning" in quotes, since nuclear fusion is unrelated to oxidation.) When the fuel is used up, the temperature declines and the star begins to shrink as gravity starts winning the struggle. Eventually this process is stopped when the electrons are pushed so close together that they resist further compression simply on the basis of the Pauli exclusion principle (no two fermions can be in the same state). The resulting object is called a **white dwarf**. If the mass is sufficiently high, however, even the electron degeneracy pressure is not enough, and the electrons will combine with the protons in a dramatic phase transition. The result is a **neutron star**, which consists of almost entirely neutrons (although the insides of neutron stars are not understood terribly well). Since the conditions at the center of a neutron star are very different from those on earth, we do not have a perfect understanding of the equation of state. Nevertheless, we believe that a sufficiently massive neutron star will itself be unable to resist the pull of gravity, and will continue to collapse. Since a fluid of neutrons is the densest material of which we can presently conceive, it is believed that the inevitable outcome of such a collapse is a black hole.

The process is summarized in the following diagram of radius vs. mass:



The point of the diagram is that, for any given mass M, the star will decrease in radius until it hits the line. White dwarfs are found between points A and B, and neutron stars between points C and D. Point B is at a height of somewhat less than 1.4 solar masses; the height of D is less certain, but probably less than 2 solar masses. The process of collapse is complicated, and during the evolution the star can lose or gain mass, so the endpoint of any given star is hard to predict. Nevertheless white dwarfs are all over the place, neutron stars are not uncommon, and there are a number of systems which are strongly believed to contain black holes. (Of course, you can't directly see the black hole. What you can see is radiation from matter accreting onto the hole, which heats up as it gets closer and emits radiation.)

We have seen that the Kruskal coordinate system provides a very useful representation of the Schwarzschild geometry. Before moving on to other types of black holes, we will introduce one more way of thinking about this spacetime, the Penrose (or Carter-Penrose, or conformal) diagram. The idea is to do a conformal transformation which brings the entire manifold onto a compact region such that we can fit the spacetime on a piece of paper.

Let's begin with Minkowski space, to see how the technique works. The metric in polar coordinates is

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.86}$$

Nothing unusual will happen to the θ, ϕ coordinates, but we will want to keep careful track of the ranges of the other two coordinates. In this case of course we have

$$
\begin{aligned}
-\infty < t < +\infty \\
0 \le r < +\infty \ .
\end{aligned} \tag{7.87}
$$

Technically the worldline $r = 0$ represents a coordinate singularity and should be covered by a different patch, but we all know what is going on so we'll just act like $r = 0$ is well-behaved.

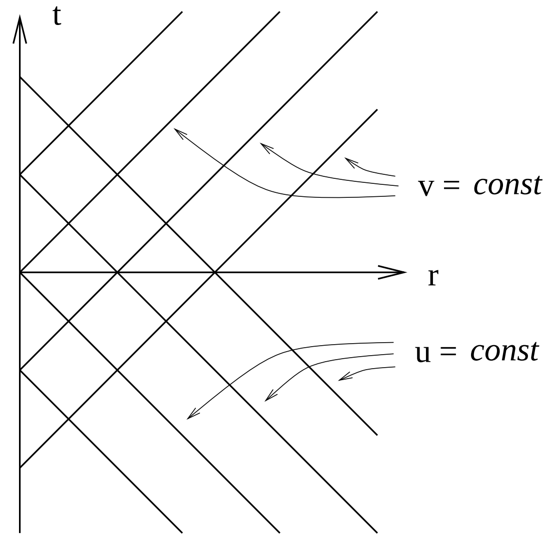
Our task is made somewhat easier if we switch to null coordinates:

$$
\begin{aligned}
u &= \frac{1}{2}(t + r) \\
v &= \frac{1}{2}(t - r) \ ,
\end{aligned} \tag{7.88}
$$

with corresponding ranges given by

$$
\begin{aligned}
-\infty < u < +\infty \\
-\infty < v < +\infty \\
v \le u \ .
\end{aligned} \tag{7.89}
$$

These ranges are as portrayed in the figure, on which each point represents a 2-sphere of

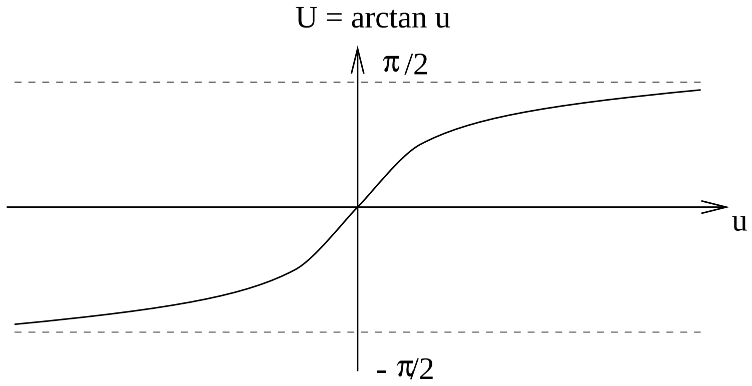


radius $r = u - v$. The metric in these coordinates is given by

$$
ds^2 = -2(\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u) + (u - v)^2 d\Omega^2 \ . \tag{7.90}
$$

We now want to change to coordinates in which "infinity" takes on a finite coordinate value. A good choice is

$$
U = \arctan u
$$



$$V = \arctan v \ . \tag{7.91}$$

The ranges are now

$$\begin{aligned} -\pi/2 < U < +\pi/2 \\ -\pi/2 < V < +\pi/2 \\ V \leq U \ . \end{aligned} \tag{7.92}$$

To get the metric, use

$$\mathrm{d}U = \frac{\mathrm{d}u}{1+u^2} \ , \tag{7.93}$$

and

$$\cos(\arctan u) = \frac{1}{\sqrt{1+u^2}} \ , \tag{7.94}$$

and likewise for v. We are led to

$$\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u = \frac{1}{\cos^2 U \cos^2 V}(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) \ . \tag{7.95}$$

Meanwhile,

$$\begin{aligned} (u-v)^2 &= (\tan U - \tan V)^2 \\ &= \frac{1}{\cos^2 U \cos^2 V}(\sin U \cos V - \cos U \sin V)^2 \\ &= \frac{1}{\cos^2 U \cos^2 V} \sin^2(U-V) \ . \end{aligned} \tag{7.96}$$

Therefore, the Minkowski metric in these coordinates is

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left[-2(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) + \sin^2(U-V)d\Omega^2\right] \ . \tag{7.97}$$

This has a certain appeal, since the metric appears as a fairly simple expression multiplied by an overall factor. We can make it even better by transforming back to a timelike coordinate η and a spacelike (radial) coordinate χ, via

$$\eta = U + V$$

$$\chi = U - V \ , \tag{7.98}$$

with ranges

$$\begin{aligned} &-\pi < \eta < +\pi \\ &0 \le \chi < +\pi \ . \end{aligned} \tag{7.99}$$

Now the metric is

$$ds^2 = \omega^{-2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi \ d\Omega^2\right) \ , \tag{7.100}$$

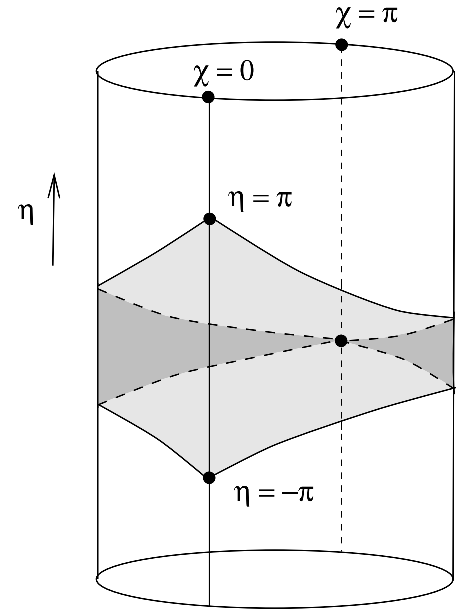
where

$$\begin{aligned} \omega &= \cos U \cos V \\ &= \frac{1}{2}(\cos\eta + \cos\chi) \ . \end{aligned} \tag{7.101}$$

The Minkowski metric may therefore be thought of as related by a conformal transformation to the "unphysical" metric

$$\begin{aligned} d\bar{s}^2 &= \omega^2 ds^2 \\ &= -\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi \ d\Omega^2 \ . \end{aligned} \tag{7.102}$$

This describes the manifold $\mathbf{R} \times S^3$, where the 3-sphere is maximally symmetric and static. There is curvature in this metric, and it is not a solution to the vacuum Einstein's equations. This shouldn't bother us, since it is unphysical; the true physical metric, obtained by a conformal transformation, is simply flat spacetime. In fact this metric is that of the "Einstein static universe," a static (but unstable) solution to Einstein's equations with a perfect fluid and a cosmological constant. Of course, the full range of coordinates on $\mathbf{R} \times S^3$ would usually be $-\infty < \eta < +\infty$, $0 \le \chi \le \pi$, while Minkowski space is mapped into the subspace defined by (7.99). The entire $\mathbf{R} \times S^3$ can be drawn as a cylinder, in which each circle is a three-sphere, as shown on the next page.



The shaded region represents Minkowski space. Note that each point (η, χ) on this cylinder is half of a two-sphere, where the other half is the point $(\eta, -\chi)$. We can unroll the shaded region to portray Minkowski space as a triangle, as shown in the figure. The is the **Penrose**

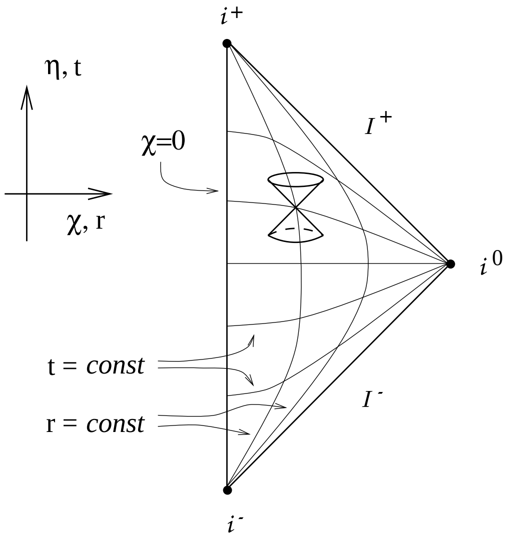


diagram. Each point represents a two-sphere.

In fact Minkowski space is only the *interior* of the above diagram (including $\chi = 0$); the boundaries are not part of the original spacetime. Together they are referred to as **conformal infinity**. The structure of the Penrose diagram allows us to subdivide conformal infinity into a few different regions:

$$\begin{aligned}
i^+ &= \text{future timelike infinity } (\eta = \pi\ ,\ \chi = 0) \\
i^0 &= \text{spatial infinity } (\eta = 0\ ,\ \chi = \pi) \\
i^- &= \text{past timelike infinity } (\eta = -\pi\ ,\ \chi = 0) \\
\mathcal{I}^+ &= \text{future null infinity } (\eta = \pi - \chi\ ,\ 0 < \chi < \pi) \\
\mathcal{I}^- &= \text{past null infinity } (\eta = -\pi + \chi\ ,\ 0 < \chi < \pi)
\end{aligned}$$

($\mathcal{I}^+$ and $\mathcal{I}^-$ are pronounced as "scri-plus" and "scri-minus", respectively.) Note that i^+, i^0, and i^- are actually *points*, since $\chi = 0$ and $\chi = \pi$ are the north and south poles of S^3. Meanwhile $\mathcal{I}^+$ and $\mathcal{I}^-$ are actually null surfaces, with the topology of $\mathbf{R} \times S^2$.

There are a number of important features of the Penrose diagram for Minkowski spacetime. The points i^+, and i^- can be thought of as the limits of spacelike surfaces whose normals are timelike; conversely, i^0 can be thought of as the limit of timelike surfaces whose normals are spacelike. Radial null geodesics are at $\pm 45°$ in the diagram. All timelike geodesics begin at i^- and end at i^+; all null geodesics begin at $\mathcal{I}^-$ and end at $\mathcal{I}^+$; all spacelike geodesics both begin and end at i^0. On the other hand, there can be non-geodesic timelike curves that end at null infinity (if they become "asymptotically null").

It is nice to be able to fit all of Minkowski space on a small piece of paper, but we don't really learn much that we didn't already know. Penrose diagrams are more useful when we want to represent slightly more interesting spacetimes, such as those for black holes. The original use of Penrose diagrams was to compare spacetimes to Minkowski space "at infinity" — the rigorous definition of "asymptotically flat" is basically that a spacetime has a conformal infinity just like Minkowski space. We will not pursue these issues in detail, but instead turn directly to analysis of the Penrose diagram for a Schwarzschild black hole.

We will not go through the necessary manipulations in detail, since they parallel the Minkowski case with considerable additional algebraic complexity. We would start with the null version of the Kruskal coordinates, in which the metric takes the form

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(\mathrm{d}u'\mathrm{d}v' + \mathrm{d}v'\mathrm{d}u') + r^2 d\Omega^2\ , \tag{7.103}$$

where r is defined implicitly via

$$u'v' = \left(\frac{r}{2GM} - 1\right) e^{r/2GM}\ . \tag{7.104}$$

Then essentially the same transformation as was used in flat spacetime suffices to bring infinity into finite coordinate values:

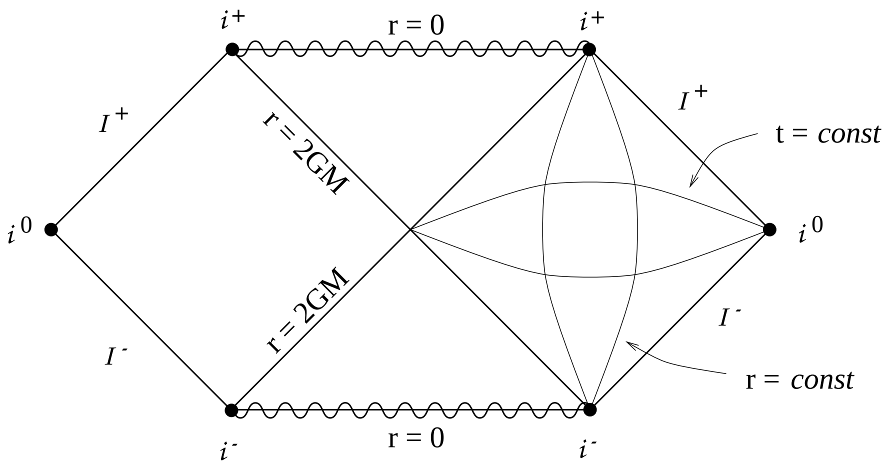
$$u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right)$$

$$v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right) , \tag{7.105}$$

with ranges

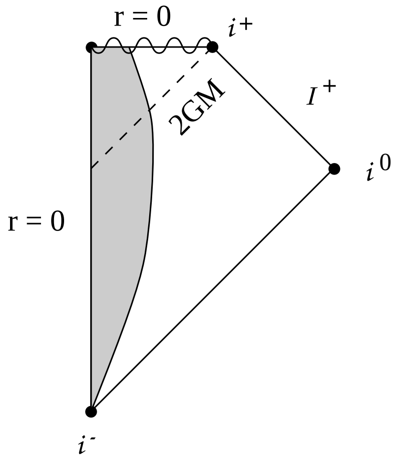
$$-\pi/2 < u'' < +\pi/2$$
$$-\pi/2 < v'' < +\pi/2$$
$$-\pi < u'' + v'' < \pi \ .$$

The (u'', v'') part of the metric (that is, at constant angular coordinates) is now conformally related to Minkowski space. In the new coordinates the singularities at $r = 0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other. The Penrose diagram for the maximally extended Schwarzschild solution thus looks like this:



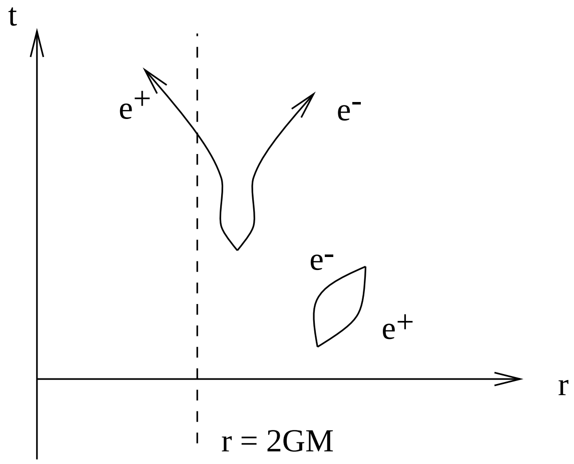
The only real subtlety about this diagram is the necessity to understand that i^+ and i^- are distinct from $r = 0$ (there are plenty of timelike paths that do not hit the singularity). Notice also that the structure of conformal infinity is just like that of Minkowski space, consistent with the claim that Schwarzschild is asymptotically flat. Also, the Penrose diagram for a collapsing star that forms a black hole is what you might expect, as shown on the next page.

Once again the Penrose diagrams for these spacetimes don't really tell us anything we didn't already know; their usefulness will become evident when we consider more general black holes. In principle there could be a wide variety of types of black holes, depending on the process by which they were formed. Surprisingly, however, this turns out not to be the case; no matter how a black hole is formed, it settles down (fairly quickly) into a state which is characterized only by the mass, charge, and angular momentum. This property, which must be demonstrated individually for the various types of fields which one might imagine go into the construction of the hole, is often stated as **"black holes have no hair."** You

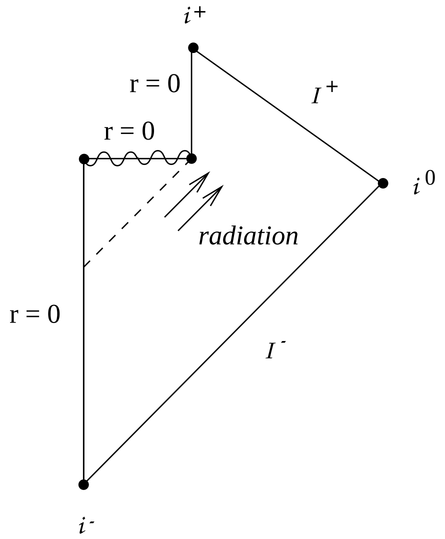


can demonstrate, for example, that a hole which is formed from an initially inhomogeneous collapse "shakes off" any lumpiness by emitting gravitational radiation. This is an example of a "no-hair theorem." If we are interested in the form of the black hole after it has settled down, we thus need only to concern ourselves with charged and rotating holes. In both cases there exist exact solutions for the metric, which we can examine closely.

But first let's take a brief detour to the world of black hole evaporation. It is strange to think of a black hole "evaporating," but in the real world black holes aren't truly black — they radiate energy as if they were a blackbody of temperature $T = \hbar/8\pi kGM$, where M is the mass of the hole and k is Boltzmann's constant. The derivation of this effect, known as **Hawking radiation**, involves the use of quantum field theory in curved spacetime and is way outside our scope right now. The informal idea is nevertheless understandable. In quantum field theory there are "vacuum fluctuations" — the spontaneous creation and annihilation of particle/antiparticle pairs in empty space. These fluctuations are precisely analogous to the zero-point fluctuations of a simple harmonic oscillator. Normally such fluctuations are



impossible to detect, since they average out to give zero total energy (although nobody knows why; that's the cosmological constant problem). In the presence of an event horizon, though, occasionally one member of a virtual pair will fall into the black hole while its partner escapes to infinity. The particle that reaches infinity will have to have a positive energy, but the total energy is conserved; therefore the black hole has to lose mass. (If you like you can think of the particle that falls in as having a negative mass.) We see the escaping particles as Hawking radiation. It's not a very big effect, and the temperature goes down as the mass goes up, so for black holes of mass comparable to the sun it is completely negligible. Still, in principle the black hole could lose all of its mass to Hawking radiation, and shrink to nothing in the process. The relevant Penrose diagram might look like this:



On the other hand, it might not. The problem with this diagram is that "information is lost" — if we draw a spacelike surface to the past of the singularity and evolve it into the future, part of it ends up crashing into the singularity and being destroyed. As a result the radiation itself contains less information than the information that was originally in the spacetime. (This is the worse than a lack of hair on the black hole. It's one thing to think that information has been trapped inside the event horizon, but it is more worrisome to think that it has disappeared entirely.) But such a process violates the conservation of information that is implicit in both general relativity and quantum field theory, the two theories that led to the prediction. This paradox is considered a big deal these days, and there are a number of efforts to understand how the information can somehow be retrieved. A currently popular explanation relies on string theory, and basically says that black holes have a lot of hair, in the form of virtual stringy states living near the event horizon. I hope you will not be disappointed to hear that we won't look at this very closely; but you should know what the problem is and that it is an area of active research these days.

With that out of our system, we now turn to electrically charged black holes. These seem at first like reasonable enough objects, since there is certainly nothing to stop us from throwing some net charge into a previously uncharged black hole. In an astrophysical situation, however, the total amount of charge is expected to be very small, especially when compared with the mass (in terms of the relative gravitational effects). Nevertheless, charged black holes provide a useful testing ground for various thought experiments, so they are worth our consideration.

In this case the full spherical symmetry of the problem is still present; we know therefore that we can write the metric as

$$ds^2 = -e^{2\alpha(r,t)}\mathrm{d}t^2 + e^{2\beta(r,t)}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{7.106}$$

Now, however, we are no longer in vacuum, since the hole will have a nonzero electromagnetic field, which in turn acts as a source of energy-momentum. The energy-momentum tensor for electromagnetism is given by

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \ , \tag{7.107}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor. Since we have spherical symmetry, the most general field strength tensor will have components

$$\begin{aligned} F_{tr} &= f(r,t) = -F_{rt} \\ F_{\theta\phi} &= g(r,t)\sin\theta = -F_{\phi\theta} \ , \end{aligned} \tag{7.108}$$

where $f(r,t)$ and $g(r,t)$ are some functions to be determined by the field equations, and components not written are zero. F_{tr} corresponds to a radial electric field, while $F_{\theta\phi}$ corresponds to a radial magnetic field. (For those of you wondering about the $\sin\theta$, recall that the thing which should be independent of θ and ϕ is the radial component of the magnetic field, $B^r = \epsilon^{01\mu\nu}F_{\mu\nu}$. For a spherically symmetric metric, $\epsilon^{\rho\sigma\mu\nu} = \frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\rho\sigma\mu\nu}$ is proportional to $(\sin\theta)^{-1}$, so we want a factor of $\sin\theta$ in $F_{\theta\phi}$.) The field equations in this case are both Einstein's equations and Maxwell's equations:

$$\begin{aligned} g^{\mu\nu}\nabla_\mu F_{\nu\sigma} &= 0 \\ \nabla_{[\mu}F_{\nu\rho]} &= 0 \ . \end{aligned} \tag{7.109}$$

The two sets are coupled together, since the electromagnetic field strength tensor enters Einstein's equations through the energy-momentum tensor, while the metric enters explicitly into Maxwell's equations.

The difficulties are not insurmountable, however, and a procedure similar to the one we followed for the vacuum case leads to a solution for the charged case as well. We will not go through the steps explicitly, but merely quote the final answer. The solution is known as the **Reissner-Nordstrøm metric**, and is given by

$$ds^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 , \tag{7.110}$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2} . \tag{7.111}$$

In this expression, M is once again interpreted as the mass of the hole; q is the total electric charge, and p is the total magnetic charge. Isolated magnetic charges (monopoles) have never been observed in nature, but that doesn't stop us from writing down the metric that they would produce if they did exist. There are good theoretical reasons to think that monopoles exist, but are extremely rare. (Of course, there is also the possibility that a black hole could have magnetic charge even if there aren't any monopoles.) In fact the electric and magnetic charges enter the metric in the same way, so we are not introducing any additional complications by keeping p in our expressions. The electromagnetic fields associated with this solution are given by

$$\begin{aligned} F_{tr} &= -\frac{q}{r^2} \\ F_{\theta\phi} &= p \sin\theta . \end{aligned} \tag{7.112}$$

Conservatives are welcome to set $p = 0$ if they like.

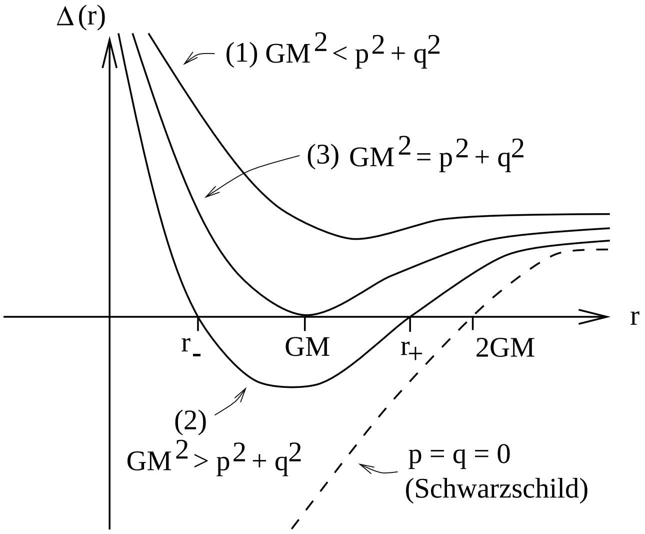
The structure of singularities and event horizons is more complicated in this metric than it was in Schwarzschild, due to the extra term in the function $\Delta(r)$ (which can be thought of as measuring "how much the light cones tip over"). One thing remains the same: at $r = 0$ there is a true curvature singularity (as could be checked by computing the curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$). Meanwhile, the equivalent of $r = 2GM$ will be the radius where Δ vanishes. This will occur at

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(p^2+q^2)} . \tag{7.113}$$

This might constitute two, one, or zero solutions, depending on the relative values of GM^2 and $p^2 + q^2$. We therefore consider each case separately.

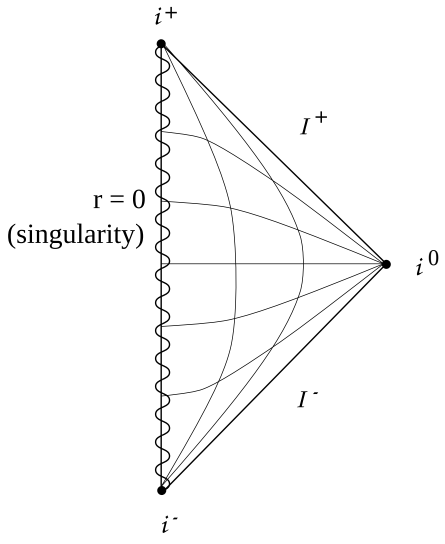
Case One — $GM^2 < p^2 + q^2$

In this case the coefficient Δ is always positive (never zero), and the metric is completely regular in the (t, r, θ, ϕ) coordinates all the way down to $r = 0$. The coordinate t is always timelike, and r is always spacelike. But there still is the singularity at $r = 0$, which is now a timelike line. Since there is no event horizon, there is no obstruction to an observer travelling to the singularity and returning to report on what was observed. This is known as a **naked singularity**, one which is not shielded by an horizon. A careful analysis of the geodesics



reveals, however, that the singularity is "repulsive" — timelike geodesics never intersect $r = 0$, instead they approach and then reverse course and move away. (Null geodesics can reach the singularity, as can non-geodesic timelike curves.)

As $r \to \infty$ the solution approaches flat spacetime, and as we have just seen the causal structure is "normal" everywhere. The Penrose diagram will therefore be just like that of Minkowski space, except that now $r = 0$ is a singularity.



The nakedness of the singularity offends our sense of decency, as well as the **cosmic censorship conjecture**, which roughly states that the gravitational collapse of physical matter configurations will never produce a naked singularity. (Of course, it's just a conjecture, and it may not be right; there are some claims from numerical simulations that collapse of spindle-like configurations can lead to naked singularities.) In fact, we should not ever expect to find a black hole with $GM^2 < p^2 + q^2$ as the result of gravitational collapse. Roughly speaking, this condition states that the total energy of the hole is less than the contribution to the energy from the electromagnetic fields alone — that is, the mass of the matter which carried the charge would have had to be negative. This solution is therefore generally considered to be unphysical. Notice also that there are not good Cauchy surfaces (spacelike slices for which every inextendible timelike line intersects them) in this spacetime, since timelike lines can begin and end at the singularity.

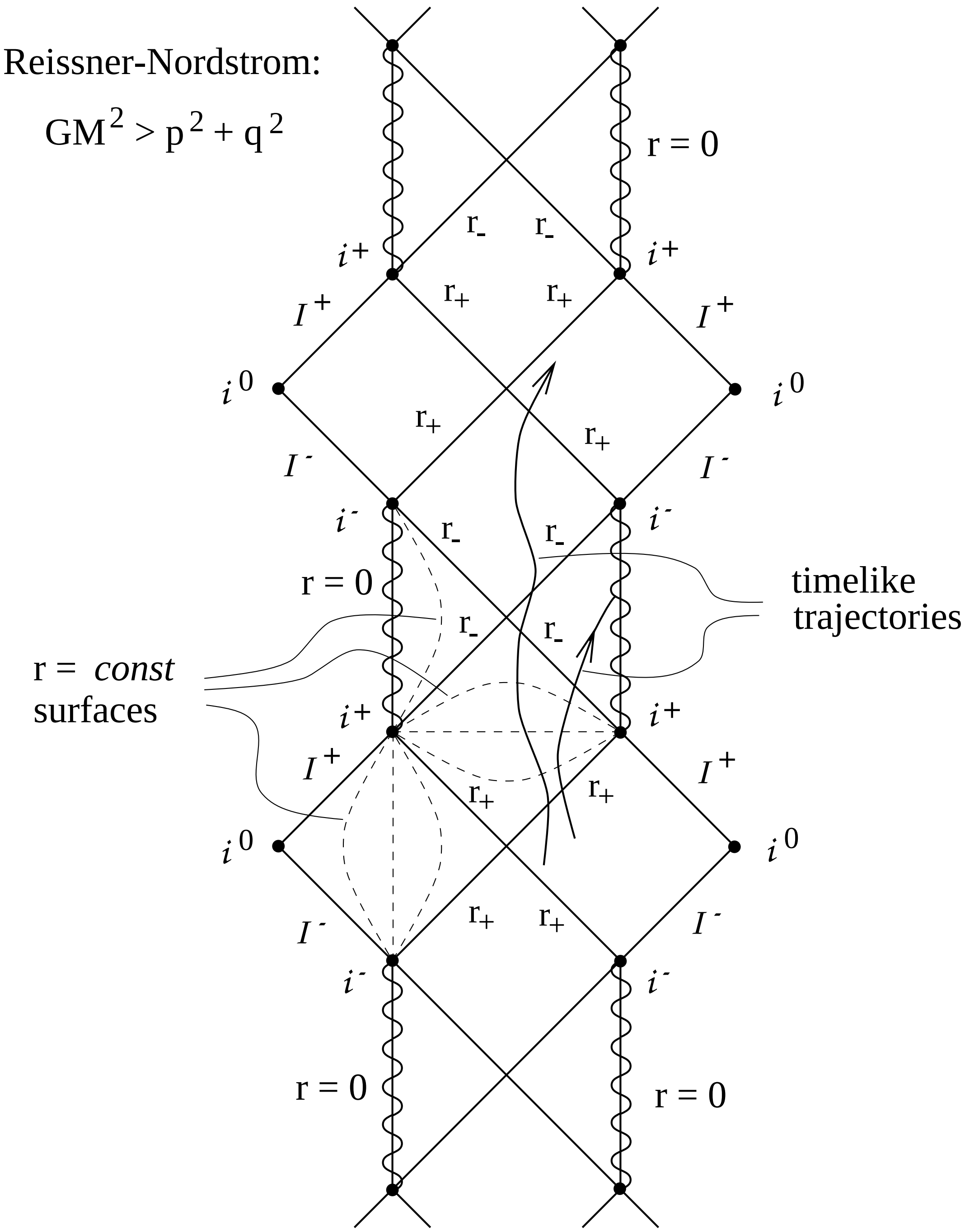
Case Two — $GM^2 > p^2 + q^2$

This is the situation which we expect to apply in real gravitational collapse; the energy in the electromagnetic field is less than the total energy. In this case the metric coefficient $\Delta(r)$ is positive at large r and small r, and negative inside the two vanishing points $r_\pm = GM \pm \sqrt{G^2M^2 - G(p^2+q^2)}$. The metric has coordinate singularities at both r_+ and r_-; in both cases these could be removed by a change of coordinates as we did with Schwarzschild.

The surfaces defined by $r = r_\pm$ are both null, and in fact they are event horizons (in a sense we will make precise in a moment). The singularity at $r = 0$ is a timelike line (not a spacelike surface as in Schwarzschild). If you are an observer falling into the black hole from far away, r_+ is just like $2GM$ in the Schwarzschild metric; at this radius r switches from being a spacelike coordinate to a timelike coordinate, and you necessarily move in the direction of decreasing r. Witnesses outside the black hole also see the same phenomena that they would outside an uncharged hole — the infalling observer is seen to move more and more slowly, and is increasingly redshifted.

But the inevitable fall from r_+ to ever-decreasing radii only lasts until you reach the null surface $r = r_-$, where r switches back to being a spacelike coordinate and the motion in the direction of decreasing r can be arrested. Therefore you do not have to hit the singularity at $r = 0$; this is to be expected, since $r = 0$ is a timelike line (and therefore not necessarily in your future). In fact you can choose either to continue on to $r = 0$, or begin to move in the direction of increasing r back through the null surface at $r = r_-$. Then r will once again be a timelike coordinate, but with reversed orientation; you are forced to move in the direction of *increasing* r. You will eventually be spit out past $r = r_+$ once more, which is like emerging from a white hole into the rest of the universe. From here you can choose to go back into the black hole — this time, a different hole than the one you entered in the first place — and repeat the voyage as many times as you like. This little story corresponds to the accompanying Penrose diagram, which of course can be derived more rigorously by choosing appropriate coordinates and analytically extending the Reissner-Nordstrøm metric as far as it will go.

How much of this is science, as opposed to science fiction? Probably not much. If you think about the world as seen from an observer inside the black hole who is about to cross the event horizon at r_-, you will notice that they can look back in time to see the entire history

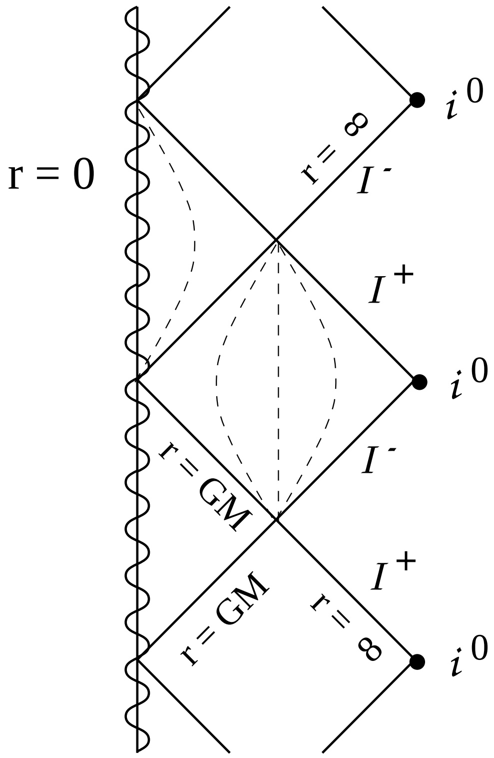


Reissner-Nordstrom:
GM² > p² + q²
r = 0
r-
r-
i+
i+
r+
r+
I+
I+
i0
i0
r+
r+
I-
I-
i-
i-
r-
r-
r = 0
timelike
trajectories
r-
r-
r = const
surfaces
i+
i+
I+
I+
r+
r+
i0
i0
r+
r+
I-
I-
i-
i-
r = 0
r = 0

of the external (asymptotically flat) universe, at least as seen from the black hole. But they see this (infinitely long) history in a finite amount of their proper time — thus, any signal that gets to them as they approach r_- is infinitely blueshifted. Therefore it is reasonable to believe (although I know of no proof) that any non-spherically symmetric perturbation that comes into a Reissner-Nordstrøm black hole will violently disturb the geometry we have described. It's hard to say what the actual geometry will look like, but there is no very good reason to believe that it must contain an infinite number of asymptotically flat regions connecting to each other via various wormholes.

Case Three — $GM^2 = p^2 + q^2$

This case is known as the **extreme** Reissner-Nordstrøm solution (or simply "extremal black hole"). The mass is exactly balanced in some sense by the charge — you can construct exact solutions consisting of several extremal black holes which remain stationary with respect to each other for all time. On the one hand the extremal hole is an amusing theoretical toy; these solutions are often examined in studies of the information loss paradox, and the role of black holes in quantum gravity. On the other hand it appears very unstable, since adding just a little bit of matter will bring it to Case Two.



The extremal black holes have $\Delta(r) = 0$ at a single radius, $r = GM$. This does represent an event horizon, but the r coordinate is never timelike; it becomes null at $r = GM$, but is spacelike on either side. The singularity at $r = 0$ is a timelike line, as in the other cases. So for this black hole you can again avoid the singularity and continue to move to the future to extra copies of the asymptotically flat region, but the singularity is always "to the left." The Penrose diagram is as shown.

We could of course go into a good deal more detail about the charged solutions, but let's instead move on to spinning black holes. It is much more difficult to find the exact solution for the metric in this case, since we have given up on spherical symmetry. To begin with all that is present is axial symmetry (around the axis of rotation), but we can also ask for stationary solutions (a timelike Killing vector). Although the Schwarzschild and Reissner-Nordstrøm solutions were discovered soon after general relativity was invented, the solution for a rotating black hole was found by Kerr only in 1963. His result, the **Kerr metric**, is given by the following mess:

$$ds^2 = -\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t)^2\ , \qquad (7.114)$$

where

$$\Delta(r) = r^2 - 2GMr + a^2\ , \qquad (7.115)$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta\ . \qquad (7.116)$$

Here a measures the rotation of the hole and M is the mass. It is straightforward to include electric and magnetic charges q and p, simply by replacing $2GMr$ with $2GMr-(q^2+p^2)/G$; the result is the **Kerr-Newman metric**. All of the interesting phenomena persist in the absence of charges, so we will set $q=p=0$ from now on.

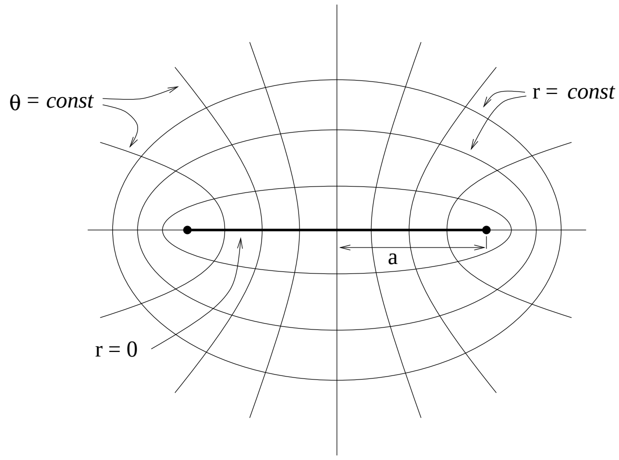
The coordinates (t,r,θ,ϕ) are known as **Boyer-Lindquist coordinates**. It is straightforward to check that as $a\to 0$ they reduce to Schwarzschild coordinates. If we keep a fixed and let $M\to 0$, however, we recover flat spacetime but not in ordinary polar coordinates. The metric becomes

$$ds^2 = -\mathrm{d}t^2 + \frac{(r^2+a^2\cos^2\theta)^2}{(r^2+a^2)}\mathrm{d}r^2 + (r^2+a^2\cos^2\theta)^2\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2\ , \qquad (7.117)$$

and we recognize the spatial part of this as flat space in ellipsoidal coordinates.
They are related to Cartesian coordinates in Euclidean 3-space by

$$\begin{aligned} x &= (r^2+a^2)^{1/2}\sin\theta\ \cos(\phi) \\ y &= (r^2+a^2)^{1/2}\sin\theta\ \sin(\phi) \\ z &= r\cos\theta\ . \end{aligned} \qquad (7.118)$$

There are two Killing vectors of the metric (7.114), both of which are manifest; since the metric coefficients are independent of t and ϕ, both $\zeta^\mu = \partial_t$ and $\eta^\mu = \partial_\phi$ are Killing vectors.



Of course η^μ expresses the axial symmetry of the solution. The vector ζ^μ is not orthogonal to $t =$ constant hypersurfaces, and in fact is not orthogonal to any hypersurfaces at all; hence this metric is stationary, but not static. (It's not changing with time, but it is spinning.)

What is more, the Kerr metric also possesses something called a **Killing tensor**. This is any symmetric $(0, n)$ tensor $\xi_{\mu_1\cdots\mu_n}$ which satisfies

$$\nabla_{(\sigma}\xi_{\mu_1\cdots\mu_n)} = 0 \ . \tag{7.119}$$

Simple examples of Killing tensors are the metric itself, and symmetrized tensor products of Killing vectors. Just as a Killing vector implies a constant of geodesic motion, if there exists a Killing tensor then along a geodesic we will have

$$\xi_{\mu_1\cdots\mu_n}\frac{dx^{\mu_1}}{d\lambda}\cdots\frac{dx^{\mu_n}}{d\lambda} = \text{constant} \ . \tag{7.120}$$

(Unlike Killing vectors, higher-rank Killing tensors do not correspond to symmetries of the metric.) In the Kerr geometry we can define the $(0, 2)$ tensor

$$\xi_{\mu\nu} = 2\rho^2 l_{(\mu}n_{\nu)} + r^2 g_{\mu\nu} \ . \tag{7.121}$$

In this expression the two vectors l and n are given (with indices raised) by

$$\begin{aligned} l^\mu &= \frac{1}{\Delta}\left(r^2 + a^2, \Delta, 0, a\right) \\ n^\mu &= \frac{1}{2\rho^2}\left(r^2 + a^2, -\Delta, 0, a\right) \ . \end{aligned} \tag{7.122}$$

Both vectors are null and satisfy

$$l^\mu l_\mu = 0 \ , \quad n^\mu n_\mu = 0 \ , \quad l^\mu n_\mu = -1 \ . \tag{7.123}$$

(For what it is worth, they are the "special null vectors" of the Petrov classification for this spacetime.) With these definitions, you can check for yourself that $\xi_{\mu\nu}$ is a Killing tensor.

Let's think about the structure of the full Kerr solution. Singularities seem to appear at both $\Delta = 0$ and $\rho = 0$; let's turn our attention first to $\Delta = 0$. As in the Reissner-Nordstrøm solution there are three possibilities: $G^2M^2 > a^2$, $G^2M^2 = a^2$, and $G^2M^2 < a^2$. The last case features a naked singularity, and the extremal case $G^2M^2 = a^2$ is unstable, just as in Reissner-Nordstrøm. Since these cases are of less physical interest, and time is short, we will concentrate on $G^2M^2 > a^2$. Then there are two radii at which Δ vanishes, given by

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \ . \tag{7.124}$$

Both radii are null surfaces which will turn out to be event horizons. The analysis of these surfaces proceeds in close analogy with the Reissner-Nordstrøm case; it is straightforward to find coordinates which extend through the horizons.

Besides the event horizons at $r_\pm$, the Kerr solution also features an additional surface of interest. Recall that in the spherically symmetric solutions, the "timelike" Killing vector $\zeta^\mu = \partial_t$ actually became null on the (outer) event horizon, and spacelike inside. Checking to see where the analogous thing happens for Kerr, we compute

$$\zeta^\mu \zeta_\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \ . \tag{7.125}$$

This does not vanish at the outer event horizon; in fact, at $r = r_+$ (where $\Delta = 0$), we have

$$\zeta^\mu \zeta_\mu = \frac{a^2}{\rho^2} \sin^2\theta \geq 0 \ . \tag{7.126}$$

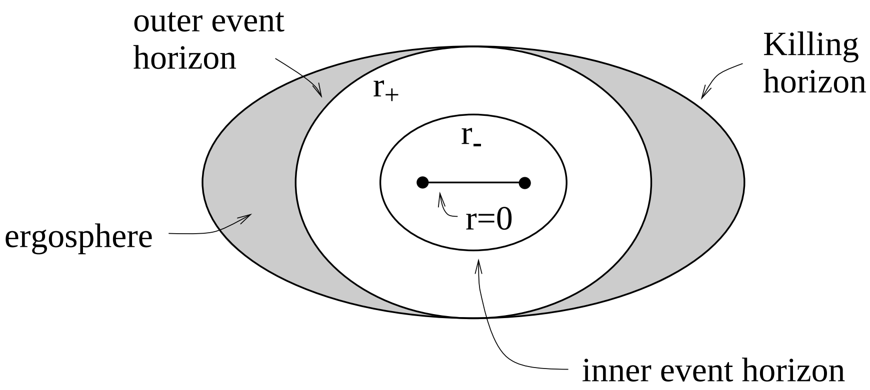
So the Killing vector is already spacelike at the outer horizon, except at the north and south poles ($\theta = 0$) where it is null. The locus of points where $\zeta^\mu \zeta_\mu = 0$ is known as the **Killing horizon**, and is given by

$$(r - GM)^2 = G^2M^2 - a^2 \cos^2\theta \ , \tag{7.127}$$

while the outer event horizon is given by

$$(r_+ - GM)^2 = G^2M^2 - a^2 \ . \tag{7.128}$$

There is thus a region in between these two surfaces, known as the **ergosphere**. Inside the ergosphere, you must move in the direction of the rotation of the black hole (the ϕ direction); however, you can still towards or away from the event horizon (and there is no trouble exiting the ergosphere). It is evidently a place where interesting things can happen even before you cross the horizon; more details on this later.



Before rushing to draw Penrose diagrams, we need to understand the nature of the true curvature singularity; this does not occur at $r = 0$ in this spacetime, but rather at $\rho = 0$. Since $\rho^2 = r^2 + a^2 \cos^2\theta$ is the sum of two manifestly nonnegative quantities, it can only vanish when both quantities are zero, or

$$r = 0 \ , \qquad \theta = \frac{\pi}{2} \ . \tag{7.129}$$

This seems like a funny result, but remember that $r = 0$ is not a point in space, but a disk; the set of points $r = 0$, $\theta = \pi/2$ is actually the *ring* at the edge of this disk. The rotation has "softened" the Schwarzschild singularity, spreading it out over a ring.

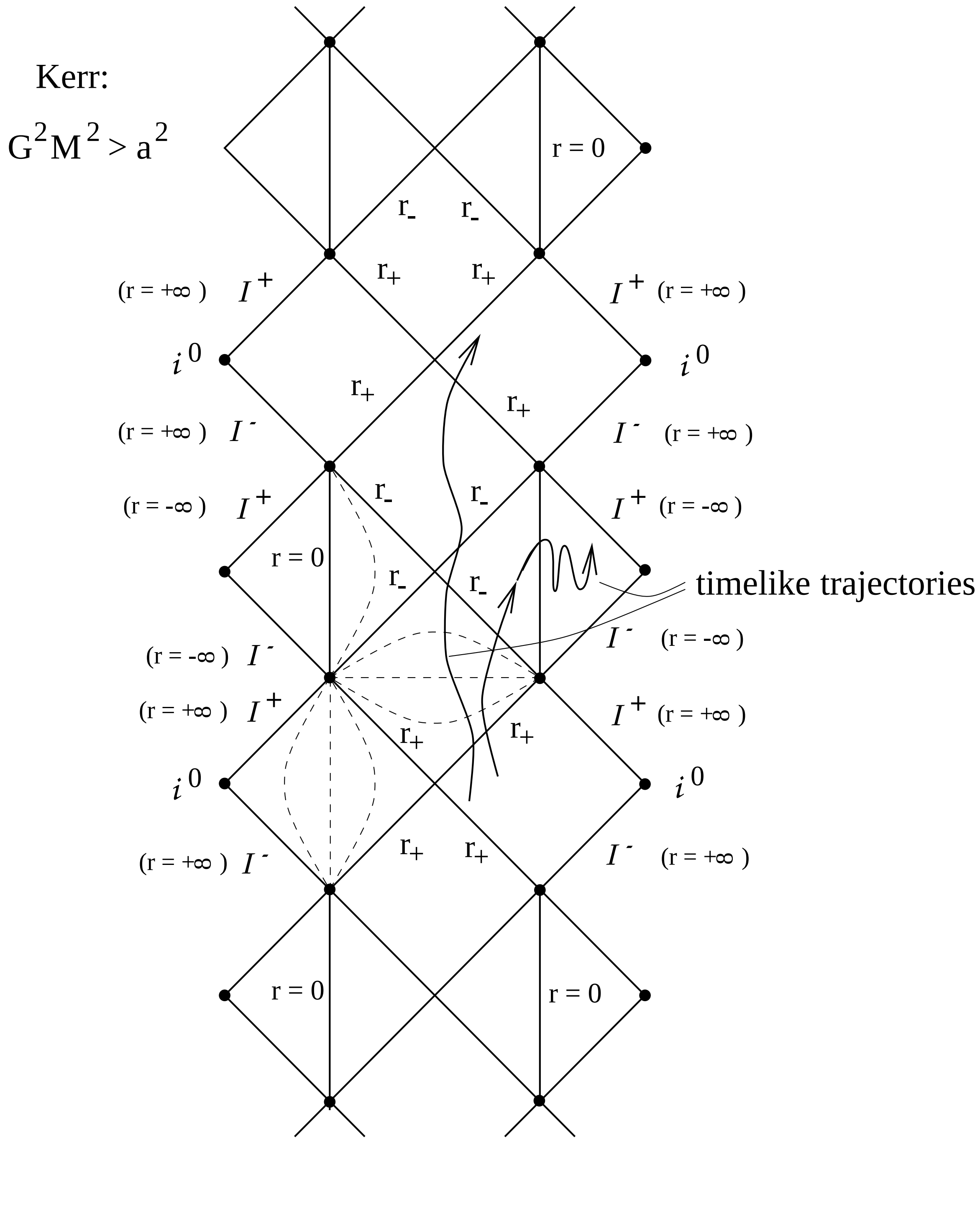
What happens if you go inside the ring? A careful analytic continuation (which we will not perform) would reveal that you exit to another asymptotically flat spacetime, but not an identical copy of the one you came from. The new spacetime is described by the Kerr metric with $r < 0$. As a result, Δ never vanishes and there are no horizons. The Penrose diagram is much like that for Reissner-Nordstrøm, except now you can pass through the singularity.

Not only do we have the usual strangeness of these distinct asymptotically flat regions connected to ours through the black hole, but the region near the ring singularity has additional pathologies: closed timelike curves. If you consider trajectories which wind around in ϕ while keeping θ and t constant and r a small negative value, the line element along such a path is

$$ds^2 = a^2\left(1 + \frac{2GM}{r}\right) \mathrm{d}\phi^2 \ , \tag{7.130}$$

which is negative for small negative r. Since these paths are closed, they are obviously CTC's. You can therefore meet yourself in the past, with all that entails.

Of course, everything we say about the analytic extension of Kerr is subject to the same caveats we mentioned for Schwarzschild and Reissner-Nordstrøm; it is unlikely that realistic gravitational collapse leads to these bizarre spacetimes. It is nevertheless always useful to have exact solutions. Furthermore, for the Kerr metric there are strange things happening even if we stay outside the event horizon, to which we now turn.


Kerr:
$G^2M^2 > a^2$
r = 0
r_- r_-
r_+ r_+
(r = +∞) I^+
I^+ (r = +∞)
i^0
i^0
r_+
r_+
(r = +∞) I^-
I^- (r = +∞)
r_- r_-
(r = -∞) I^+
I^+ (r = -∞)
r = 0
r_- r_-
timelike trajectories
(r = -∞) I^-
I^- (r = -∞)
(r = +∞) I^+
I^+ (r = +∞)
r_+ r_+
i^0
i^0
r_+ r_+
(r = +∞) I^-
I^- (r = +∞)
r = 0
r = 0

We begin by considering more carefully the angular velocity of the hole. Obviously the conventional definition of angular velocity will have to be modified somewhat before we can apply it to something as abstract as the metric of spacetime. Let us consider the fate of a photon which is emitted in the ϕ direction at some radius r in the equatorial plane ($\theta = \pi/2$) of a Kerr black hole. The instant it is emitted its momentum has no components in the r or θ direction, and therefore the condition that it be null is

$$ds^2 = 0 = g_{tt}\mathrm{d}t^2 + g_{t\phi}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) + g_{\phi\phi}\mathrm{d}\phi^2 \ . \tag{7.131}$$

This can be immediately solved to obtain

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \ . \tag{7.132}$$

If we evaluate this quantity on the Killing horizon of the Kerr metric, we have $g_{tt} = 0$, and the two solutions are

$$\frac{d\phi}{dt} = 0 \ , \qquad \frac{d\phi}{dt} = \frac{2a}{(2GM)^2 + a^2} \ . \tag{7.133}$$

The nonzero solution has the same sign as a; we interpret this as the photon moving around the hole in the same direction as the hole's rotation. The zero solution means that the photon directed against the hole's rotation doesn't move at all in this coordinate system. (This isn't a full solution to the photon's trajectory, just the statement that its instantaneous velocity is zero.) This is an example of the "dragging of inertial frames" mentioned earlier. The point of this exercise is to note that massive particles, which must move more slowly than photons, are necessarily dragged along with the hole's rotation once they are inside the Killing horizon. This dragging continues as we approach the outer event horizon at r_+; we can define the angular velocity of the event horizon itself, Ω_H, to be the minimum angular velocity of a particle at the horizon. Directly from (7.132) we find that

$$\Omega_H = \left(\frac{d\phi}{dt}\right)_-(r_+) = \frac{a}{r_+^2 + a^2} \ . \tag{7.134}$$

Now let's turn to geodesic motion, which we know will be simplified by considering the conserved quantities associated with the Killing vectors $\zeta^\mu = \partial_t$ and $\eta^\mu = \partial_\phi$. For the purposes at hand we can restrict our attention to massive particles, for which we can work with the four-momentum

$$p^\mu = m\frac{dx^\mu}{d\tau} \ , \tag{7.135}$$

where m is the rest mass of the particle. Then we can take as our two conserved quantities the actual energy and angular momentum of the particle,

$$E = -\zeta_\mu p^\mu = m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2mGMar}{\rho^2}\sin^2\theta\,\frac{d\phi}{d\tau} \tag{7.136}$$

and

$$L = \eta_\mu p^\mu = -\frac{2mGMar}{\rho^2}\sin^2\theta\,\frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta\,\frac{d\phi}{d\tau}\ . \tag{7.137}$$

(These differ from our previous definitions for the conserved quantities, where E and L were taken to be the energy and angular momentum *per unit mass*. They are conserved either way, of course.)

The minus sign in the definition of E is there because at infinity both ζ^μ and p^μ are timelike, so their inner product is negative, but we want the energy to be positive. Inside the ergosphere, however, ζ^μ becomes spacelike; we can therefore imagine particles for which

$$E = -\zeta_\mu p^\mu < 0\ . \tag{7.138}$$

The extent to which this bothers us is ameliorated somewhat by the realization that *all* particles outside the Killing horizon must have positive energies; therefore a particle inside the ergosphere with negative energy must either remain on a geodesic inside the Killing horizon, or be accelerated until its energy is positive if it is to escape.

Still, this realization leads to a way to extract energy from a rotating black hole; the method is known as the **Penrose process**. The idea is simple; starting from outside the ergosphere, you arm yourself with a large rock and leap toward the black hole. If we call the four-momentum of the (you + rock) system $p^{(0)\mu}$, then the energy $E^{(0)} = -\zeta_\mu p^{(0)\mu}$ is certainly positive, and conserved as you move along your geodesic. Once you enter the ergosphere, you hurl the rock with all your might, in a very specific way. If we call your momentum $p^{(1)\mu}$ and that of the rock $p^{(2)\mu}$, then at the instant you throw it we have conservation of momentum just as in special relativity:

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}\ . \tag{7.139}$$

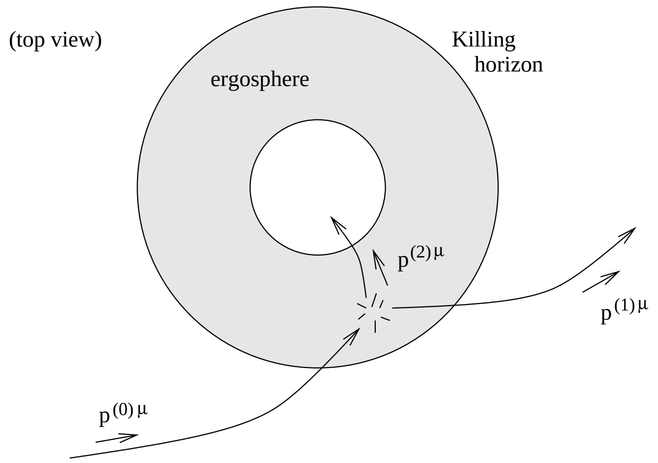
Contracting with the Killing vector ζ_μ gives

$$E^{(0)} = E^{(1)} + E^{(2)}\ . \tag{7.140}$$

But, if we imagine that you are arbitrarily strong (and accurate), you can arrange your throw such that $E^{(2)} < 0$, as per (7.158). Furthermore, Penrose was able to show that you can arrange the initial trajectory and the throw such that afterwards you follow a geodesic trajectory back outside the Killing horizon into the external universe. Since your energy is conserved along the way, at the end we will have

$$E^{(1)} > E^{(0)}\ . \tag{7.141}$$

Thus, you have emerged with *more* energy than you entered with.



There is no such thing as a free lunch; the energy you gained came from somewhere, and that somewhere is the black hole. In fact, the Penrose process extracts energy from the rotating black hole by decreasing its angular momentum; you have to throw the rock against the hole's rotation to get the trick to work. To see this more precisely, define a new Killing vector

$$\chi^\mu = \zeta^\mu + \Omega_H \eta^\mu \ . \tag{7.142}$$

On the outer horizon χ^μ is null and tangent to the horizon. (This can be seen from $\zeta^\mu = \partial_t$, $\eta^\mu = \partial_\phi$, and the definition (7.134) of Ω_H.) The statement that the particle with momentum $p^{(2)\mu}$ crosses the event horizon "moving forwards in time" is simply

$$p^{(2)\mu}\chi_\mu < 0 \ . \tag{7.143}$$

Plugging in the definitions of E and L, we see that this condition is equivalent to

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H} \ . \tag{7.144}$$

Since we have arranged $E^{(2)}$ to be negative, and Ω_H is positive, we see that the particle must have a negative angular momentum — it is moving against the hole's rotation. Once you have escaped the ergosphere and the rock has fallen inside the event horizon, the mass and angular momentum of the hole are what they used to be plus the negative contributions of the rock:

$$\begin{aligned} \delta M &= E^{(2)} \\ \delta J &= L^{(2)} . \end{aligned} \tag{7.145}$$

Here we have introduced the notation J for the angular momentum of the black hole; it is given by

$$J = Ma \ . \tag{7.146}$$

We won't justify this, but you can look in Wald for an explanation. Then (7.144) becomes a limit on how much you can decrease the angular momentum:

$$\delta J < \frac{\delta M}{\Omega_H} . \tag{7.147}$$

If we exactly reach this limit, as the rock we throw in becomes more and more null, we have the "ideal" process, in which $\delta J = \delta M/\Omega_H$.

We will now use these ideas to prove a powerful result: although you can use the Penrose process to extract energy from the black hole, you can never decrease the area of the event horizon. For a Kerr metric, one can go through a straightforward computation (projecting the metric and volume element and so on) to compute the area of the event horizon:

$$A = 4\pi(r_+^2 + a^2) . \tag{7.148}$$

To show that this doesn't decrease, it is most convenient to work instead in terms of the **irreducible mass** of the black hole, defined by

$$\begin{aligned} M_{\rm irr}^2 &= \frac{A}{16\pi G^2} \\ &= \frac{1}{4G^2}(r_+^2 + a^2) \\ &= \frac{1}{2}\left(M^2 + \sqrt{M^4 - (Ma/G)^2}\right) \\ &= \frac{1}{2}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right) . \end{aligned} \tag{7.149}$$

We can differentiate to obtain, after a bit of work,

$$\delta M_{\rm irr} = \frac{a}{4G\sqrt{G^2M^2 - a^2}M_{\rm irr}}(\Omega_H^{-1}\delta M - \delta J) . \tag{7.150}$$

(I think I have the factors of G right, but it wouldn't hurt to check.) Then our limit (7.147) becomes

$$\delta M_{\rm irr} > 0 . \tag{7.151}$$

The irreducible mass can never be reduced; hence the name. It follows that the maximum amount of energy we can extract from a black hole before we slow its rotation to zero is

$$M - M_{\rm irr} = M - \frac{1}{\sqrt{2}}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right)^{1/2} . \tag{7.152}$$

The result of this complete extraction is a Schwarzschild black hole of mass $M_{\rm irr}$. It turns out that the best we can do is to start with an extreme Kerr black hole; then we can get out approximately 29% of its total energy.

The irreducibility of $M_{\rm irr}$ leads immediately to the fact that the area A can never decrease. From (7.149) and (7.150) we have

$$\delta A = 8\pi G \frac{a}{\Omega_H\sqrt{G^2M^2 - a^2}}(\delta M - \Omega_H \delta_J) \ , \tag{7.153}$$

which can be recast as

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J \ , \tag{7.154}$$

where we have introduced

$$\kappa = \frac{\sqrt{G^2M^2 - a^2}}{2GM(GM + \sqrt{G^2M^2 - a^2})} \ . \tag{7.155}$$

The quantity κ is known as the **surface gravity** of the black hole.

It was equations like (7.154) that first started people thinking about the relationship between black holes and thermodynamics. Consider the first law of thermodynamics,

$$dU = TdS + \text{work terms} \ . \tag{7.156}$$

It is natural to think of the term $\Omega_H \delta J$ as "work" that we do on the black hole by throwing rocks into it. Then the thermodynamic analogy begins to take shape if we think of identifying the area A as the entropy S, and the surface gravity κ as $8\pi G$ times the temperature T. In fact, in the context of classical general relativity the analogy is essentially perfect. The "zeroth" law of thermodynamics states that in thermal equilibrium the temperature is constant throughout the system; the analogous statement for black holes is that stationary black holes have constant surface gravity on the entire horizon (true). As we have seen, the first law (7.156) is equivalent to (7.154). The second law, that entropy never decreases, is simply the statement that the area of the horizon never decreases. Finally, the third law is that it is impossible to achieve $T = 0$ in any physical process, which should imply that it is impossible to achieve $\kappa = 0$ in any physical process. It turns out that $\kappa = 0$ corresponds to the extremal black holes (either in Kerr or Reissner-Nordstrøm) — where the naked singularities would appear. Somehow, then, the third law is related to cosmic censorship.

The missing piece is that *real* thermodynamic bodies don't just sit there; they give off blackbody radiation with a spectrum that depends on their temperature. Black holes, it was thought before Hawking discovered his radiation, don't do that, since they're truly black. Historically, Bekenstein came up with the idea that black holes should really be honest black bodies, including the radiation at the appropriate temperature. This annoyed Hawking, who set out to prove him wrong, and ended up proving that there would be radiation after all. So the thermodynamic analogy is even better than we had any right to expect — although it is safe to say that nobody really knows why.

8 Cosmology

Contemporary cosmological models are based on the idea that the universe is pretty much the same everywhere — a stance sometimes known as the **Copernican principle**. On the face of it, such a claim seems preposterous; the center of the sun, for example, bears little resemblance to the desolate cold of interstellar space. But we take the Copernican principle to only apply on the very largest scales, where local variations in density are averaged over. Its validity on such scales is manifested in a number of different observations, such as number counts of galaxies and observations of diffuse X-ray and γ-ray backgrounds, but is most clear in the 3° microwave background radiation. Although we now know that the microwave background is not perfectly smooth (and nobody ever expected that it was), the deviations from regularity are on the order of 10^{-5} or less, certainly an adequate basis for an approximate description of spacetime on large scales.

The Copernican principle is related to two more mathematically precise properties that a manifold might have: isotropy and homogeneity. **Isotropy** applies at some specific point in the space, and states that the space looks the same no matter what direction you look in. More formally, a manifold M is isotropic around a point p if, for any two vectors V and W in T_pM, there is an isometry of M such that the pushforward of W under the isometry is parallel with V (not pushed forward). It is isotropy which is indicated by the observations of the microwave background.

Homogeneity is the statement that the metric is the same throughout the space. In other words, given any two points p and q in M, there is an isometry which takes p into q. Note that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as $\mathbf{R} \times S^2$ in the usual metric), or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic *everywhere* then it is homogeneous. (Likewise if it is isotropic around one point and also homogeneous, it will be isotropic around every point.) Since there is ample observational evidence for isotropy, and the Copernican principle would have us believe that we are not the center of the universe and therefore observers elsewhere should also observe isotropy, we will henceforth assume both homogeneity and isotropy.

There is one catch. When we look at distant galaxies, they appear to be receding from us; the universe is apparently not static, but changing with time. Therefore we begin construction of cosmological models with the idea that the universe is homogeneous and isotropic in space, but not in time. In general relativity this translates into the statement that the universe can be foliated into spacelike slices such that each slice is homogeneous and isotropic.

We therefore consider our spacetime to be $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ represents the time direction and Σ is a homogeneous and isotropic three-manifold. The usefulness of homogeneity and isotropy is that they imply that Σ must be a maximally symmetric space. (Think of isotropy as invariance under rotations, and homogeneity as invariance under translations. Then homogeneity and isotropy together imply that a space has its maximum possible number of Killing vectors.) Therefore we can take our metric to be of the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(u)\mathrm{d}u^i\mathrm{d}u^j \ . \tag{8.1}$$

Here t is the timelike coordinate, and (u^1, u^2, u^3) are the coordinates on Σ; γ_{ij} is the maximally symmetric metric on Σ. This formula is a special case of (7.2), which we used to derive the Schwarzschild metric, except we have scaled t such that $g_{tt} = -1$. The function $a(t)$ is known as the **scale factor**, and it tells us "how big" the spacelike slice Σ is at the moment t. The coordinates used here, in which the metric is free of cross terms $\mathrm{d}t\,\mathrm{d}u^i$ and the spacelike components are proportional to a single function of t, are known as **comoving coordinates**, and an observer who stays at constant u^i is also called "comoving". Only a comoving observer will think that the universe looks isotropic; in fact on Earth we are not quite comoving, and as a result we see a dipole anisotropy in the cosmic microwave background as a result of the conventional Doppler effect.

Our interest is therefore in maximally symmetric Euclidean three-metrics γ_{ij}. We know that maximally symmetric metrics obey

$${}^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \ , \tag{8.2}$$

where k is some constant, and we put a superscript ${}^{(3)}$ on the Riemann tensor to remind us that it is associated with the three-metric γ_{ij}, not the metric of the entire spacetime. The Ricci tensor is then

$${}^{(3)}R_{jl} = 2k\gamma_{jl} \ . \tag{8.3}$$

If the space is to be maximally symmetric, then it will certainly be spherically symmetric. We already know something about spherically symmetric spaces from our exploration of the Schwarzschild solution; the metric can be put in the form

$$d\sigma^2 = \gamma_{ij}\mathrm{d}u^i\,\mathrm{d}u^j = e^{2\beta(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ . \tag{8.4}$$

The components of the Ricci tensor for such a metric can be obtained from (7.16), the Ricci tensor for a spherically symmetric spacetime, by setting $\alpha = 0$ and $\partial_0\beta = 0$, which gives

$$\begin{aligned}
{}^{(3)}R_{11} &= \frac{2}{r}\partial_1\beta \\
{}^{(3)}R_{22} &= e^{-2\beta}(r\partial_1\beta - 1) + 1 \\
{}^{(3)}R_{33} &= [e^{-2\beta}(r\partial_1\beta - 1) + 1]\sin^2\theta \ .
\end{aligned} \tag{8.5}$$

We set these proportional to the metric using (8.3), and can solve for $\beta(r)$:

$$\beta = -\frac{1}{2}\ln(1 - kr^2) \ . \tag{8.6}$$

This gives us the following metric on spacetime:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \ . \tag{8.7}$$

This is the **Robertson-Walker metric**. We have not yet made use of Einstein's equations; those will determine the behavior of the scale factor $a(t)$.

Note that the substitutions

$$\begin{array}{lcl} k & \to & \frac{k}{|k|} \\ r & \to & \sqrt{|k|}\, r \\ a & \to & \frac{a}{\sqrt{|k|}} \end{array} \tag{8.8}$$

leave (8.7) invariant. Therefore the only relevant parameter is $k/|k|$, and there are three cases of interest: $k = -1$, $k = 0$, and $k = +1$. The $k = -1$ case corresponds to constant negative curvature on Σ, and is called **open**; the $k = 0$ case corresponds to no curvature on Σ, and is called **flat**; the $k = +1$ case corresponds to positive curvature on Σ, and is called **closed**.

Let us examine each of these possibilities. For the flat case $k = 0$ the metric on Σ is

$$\begin{aligned} d\sigma^2 &= \mathrm{d}r^2 + r^2 d\Omega^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ , \end{aligned} \tag{8.9}$$

which is simply flat Euclidean space. Globally, it could describe $\mathbf{R}^3$ or a more complicated manifold, such as the three-torus $S^1 \times S^1 \times S^1$. For the closed case $k = +1$ we can define $r = \sin\chi$ to write the metric on Σ as

$$d\sigma^2 = \mathrm{d}\chi^2 + \sin^2\chi\, d\Omega^2 \ , \tag{8.10}$$

which is the metric of a three-sphere. In this case the only possible global structure is actually the three-sphere (except for the non-orientable manifold $\mathbf{R}\mathrm{P}^3$). Finally in the open $k = -1$ case we can set $r = \sinh\psi$ to obtain

$$d\sigma^2 = \mathrm{d}\psi^2 + \sinh^2\psi\, d\Omega^2 \ . \tag{8.11}$$

This is the metric for a three-dimensional space of constant negative curvature; it is hard to visualize, but think of the saddle example we spoke of in Section Three. Globally such a space could extend forever (which is the origin of the word "open"), but it could also describe a non-simply-connected compact space (so "open" is really not the most accurate description).

With the metric in hand, we can set about computing the connection coefficients and curvature tensor. Setting $\dot{a} \equiv da/dt$, the Christoffel symbols are given by

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{01} &= \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \\
\Gamma^1_{22} &= -r(1-kr^2) \qquad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\,\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta\ .
\end{aligned} \tag{8.12}$$

The nonzero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} \\
R_{22} &= r^2(a\ddot{a}+2\dot{a}^2+2k) \\
R_{33} &= r^2(a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta\ ,
\end{aligned} \tag{8.13}$$

and the Ricci scalar is then

$$R = \frac{6}{a^2}(a\ddot{a}+\dot{a}^2+k)\ . \tag{8.14}$$

The universe is not empty, so we are not interested in vacuum solutions to Einstein's equations. We will choose to model the matter and energy in the universe by a perfect fluid. We discussed perfect fluids in Section One, where they were defined as fluids which are isotropic in their rest frame. The energy-momentum tensor for a perfect fluid can be written

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu}\ , \tag{8.15}$$

where ρ and p are the energy density and pressure (respectively) as measured in the rest frame, and U^μ is the four-velocity of the fluid. It is clear that, if a fluid which is isotropic in some frame leads to a metric which is isotropic in some frame, the two frames will coincide; that is, the fluid will be at rest in comoving coordinates. The four-velocity is then

$$U^\mu = (1,0,0,0)\ , \tag{8.16}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}\ . \tag{8.17}$$

With one index raised this takes the more convenient form

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p) \ . \tag{8.18}$$

Note that the trace is given by

$$T = T^{\mu}{}_{\mu} = -\rho + 3p \ . \tag{8.19}$$

Before plugging in to Einstein's equations, it is educational to consider the zero component of the conservation of energy equation:

$$\begin{aligned} 0 &= \nabla_\mu T^{\mu}{}_0 \\ &= \partial_\mu T^{\mu}{}_0 + \Gamma^{\mu}_{\mu 0} T^0{}_0 - \Gamma^{\lambda}_{\mu 0} T^{\mu}{}_{\lambda} \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \ . \end{aligned} \tag{8.20}$$

To make progress it is necessary to choose an **equation of state**, a relationship between ρ and p. Essentially all of the perfect fluids relevant to cosmology obey the simple equation of state

$$p = w\rho \ , \tag{8.21}$$

where w is a constant independent of time. The conservation of energy equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \ , \tag{8.22}$$

which can be integrated to obtain

$$\rho \propto a^{-3(1+w)} \ . \tag{8.23}$$

The two most popular examples of cosmological fluids are known as **dust** and **radiation**. Dust is collisionless, nonrelativistic matter, which obeys $w = 0$. Examples include ordinary stars and galaxies, for which the pressure is negligible in comparison with the energy density. Dust is also known as "matter", and universes whose energy density is mostly due to dust are known as **matter-dominated**. The energy density in matter falls off as

$$\rho \propto a^{-3} \ . \tag{8.24}$$

This is simply interpreted as the decrease in the number density of particles as the universe expands. (For dust the energy density is dominated by the rest energy, which is proportional to the number density.) "Radiation" may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light that they become indistinguishable from photons (at least as far as their equation of state is concerned). Although radiation is a perfect fluid and thus has an energy-momentum tensor given by (8.15), we also know that $T_{\mu\nu}$ can be expressed in terms of the field strength as

$$T^{\mu\nu} = \frac{1}{4\pi}(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}) \ . \tag{8.25}$$

The trace of this is given by

$$T^{\mu}{}_{\mu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}(4)F^{\lambda\sigma}F_{\lambda\sigma}\right] = 0 \ . \tag{8.26}$$

But this must also equal (8.19), so the equation of state is

$$p = \frac{1}{3}\rho \ . \tag{8.27}$$

A universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation falls off as

$$\rho \propto a^{-4} \ . \tag{8.28}$$

Thus, the energy density in radiation falls off slightly faster than that in matter; this is because the number density of photons decreases in the same way as the number density of nonrelativistic particles, but individual photons also lose energy as a^{-1} as they redshift, as we will see later. (Likewise, massive but relativistic particles will lose energy as they "slow down" in comoving coordinates.) We believe that today the energy density of the universe is dominated by matter, with $\rho_{\rm mat}/\rho_{\rm rad} \sim 10^6$. However, in the past the universe was much smaller, and the energy density in radiation would have dominated at very early times.

There is one other form of energy-momentum that is sometimes considered, namely that of the vacuum itself. Introducing energy into the vacuum is equivalent to introducing a cosmological constant. Einstein's equations with a cosmological constant are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \ , \tag{8.29}$$

which is clearly the same form as the equations with no cosmological constant but an energy-momentum tensor for the vacuum,

$$T^{(\rm vac)}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu} \ . \tag{8.30}$$

This has the form of a perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} \ . \tag{8.31}$$

We therefore have $w = -1$, and the energy density is independent of a, which is what we would expect for the energy density of the vacuum. Since the energy density in matter and radiation decreases as the universe expands, if there is a nonzero vacuum energy it tends to win out over the long term (as long as the universe doesn't start contracting). If this happens, we say that the universe becomes **vacuum-dominated**.

We now turn to Einstein's equations. Recall that they can be written in the form (4.45):

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) . \tag{8.32}$$

The $\mu\nu = 00$ equation is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) , \tag{8.33}$$

and the $\mu\nu = ij$ equations give

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) . \tag{8.34}$$

(There is only one distinct equation from $\mu\nu = ij$, due to isotropy.) We can use (8.33) to eliminate second derivatives in (8.34), and do a little cleaning up to obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) , \tag{8.35}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} . \tag{8.36}$$

Together these are known as the **Friedmann equations**, and metrics of the form (8.7) which obey these equations define Friedmann-Robertson-Walker (FRW) universes.

There is a bunch of terminology which is associated with the cosmological parameters, and we will just introduce the basics here. The rate of expansion is characterized by the **Hubble parameter**,

$$H = \frac{\dot{a}}{a} . \tag{8.37}$$

The value of the Hubble parameter at the present epoch is the Hubble constant, H_0. There is currently a great deal of controversy about what its actual value is, with measurements falling in the range of 40 to 90 km/sec/Mpc. ("Mpc" stands for "megaparsec", which is 3×10^{24} cm.) Note that we have to divide $\dot{a}$ by a to get a measurable quantity, since the overall scale of a is irrelevant. There is also the **deceleration parameter**,

$$q = -\frac{a\ddot{a}}{\dot{a}^2} , \tag{8.38}$$

which measures the rate of change of the rate of expansion.

Another useful quantity is the **density parameter**,

$$\Omega = \frac{8\pi G}{3H^2}\rho$$

$$= \frac{\rho}{\rho_{\rm crit}} , \tag{8.39}$$

where the **critical density** is defined by

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G} . \tag{8.40}$$

This quantity (which will generally change with time) is called the "critical" density because the Friedmann equation (8.36) can be written

$$\Omega - 1 = \frac{k}{H^2 a^2} . \tag{8.41}$$

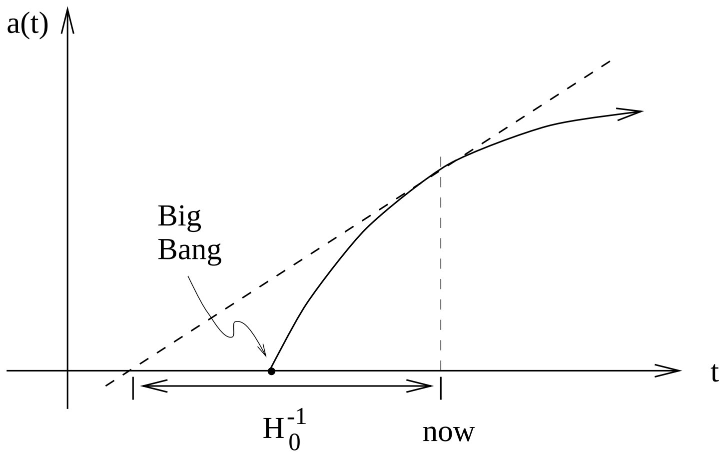
The sign of k is therefore determined by whether Ω is greater than, equal to, or less than one. We have

$$\begin{array}{ccccccc} \rho < \rho_{\rm crit} & \leftrightarrow & \Omega < 1 & \leftrightarrow & k = -1 & \leftrightarrow & \text{open} \\ \rho = \rho_{\rm crit} & \leftrightarrow & \Omega = 1 & \leftrightarrow & k = 0 & \leftrightarrow & \text{flat} \\ \rho > \rho_{\rm crit} & \leftrightarrow & \Omega > 1 & \leftrightarrow & k = +1 & \leftrightarrow & \text{closed} . \end{array}$$

The density parameter, then, tells us which of the three Robertson-Walker geometries describes our universe. Determining it observationally is an area of intense investigation.

It is possible to solve the Friedmann equations exactly in various simple cases, but it is often more useful to know the qualitative behavior of various possibilities. Let us for the moment set $\Lambda = 0$, and consider the behavior of universes filled with fluids of positive energy ($\rho > 0$) and nonnegative pressure ($p \geq 0$). Then by (8.35) we must have $\ddot{a} < 0$. Since we know from observations of distant galaxies that the universe is expanding ($\dot{a} > 0$), this means that the universe is "decelerating." This is what we should expect, since the gravitational attraction of the matter in the universe works against the expansion. The fact that the universe can only decelerate means that it must have been expanding even faster in the past; if we trace the evolution backwards in time, we necessarily reach a singularity at $a = 0$. Notice that if $\ddot{a}$ were exactly zero, $a(t)$ would be a straight line, and the age of the universe would be H_0^{-1}. Since $\ddot{a}$ is actually negative, the universe must be somewhat younger than that.

This singularity at $a = 0$ is the **Big Bang**. It represents the creation of the universe from a singular state, not explosion of matter into a pre-existing spacetime. It might be hoped that the perfect symmetry of our FRW universes was responsible for this singularity, but in fact it's not true; the singularity theorems predict that any universe with $\rho > 0$ and $p \geq 0$ must have begun at a singularity. Of course the energy density becomes arbitrarily high as $a \to 0$, and we don't expect classical general relativity to be an accurate description of nature in this regime; hopefully a consistent theory of quantum gravity will be able to fix things up.



The future evolution is different for different values of k. For the open and flat cases, $k \leq 0$, (8.36) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \ . \tag{8.42}$$

The right hand side is *strictly* positive (since we are assuming $\rho > 0$), so $\dot{a}$ never passes through zero. Since we know that today $\dot{a} > 0$, it must be positive for all time. Thus, the open and flat universes expand forever — they are temporally as well as spatially open. (Please keep in mind what assumptions go into this — namely, that there is a nonzero positive energy density. Negative energy density universes do not have to expand forever, even if they are "open".)

How fast do these universes keep expanding? Consider the quantity ρa^3 (which is constant in matter-dominated universes). By the conservation of energy equation (8.20) we have

$$\begin{aligned} \frac{d}{dt}(\rho a^3) &= a^3\left(\dot{\rho} + 3\rho\frac{\dot{a}}{a}\right) \\ &= -3pa^2\dot{a} \ . \end{aligned} \tag{8.43}$$

The right hand side is either zero or negative; therefore

$$\frac{d}{dt}(\rho a^3) \leq 0 \ . \tag{8.44}$$

This implies in turn that ρa^2 must go to zero in an ever-expanding universe, where $a \to \infty$. Thus (8.42) tells us that

$$\dot{a}^2 \to |k| \ . \tag{8.45}$$

(Remember that this is true for $k \leq 0$.) Thus, for $k = -1$ the expansion approaches the limiting value $\dot{a} \to 1$, while for $k = 0$ the universe keeps expanding, but more and more slowly.

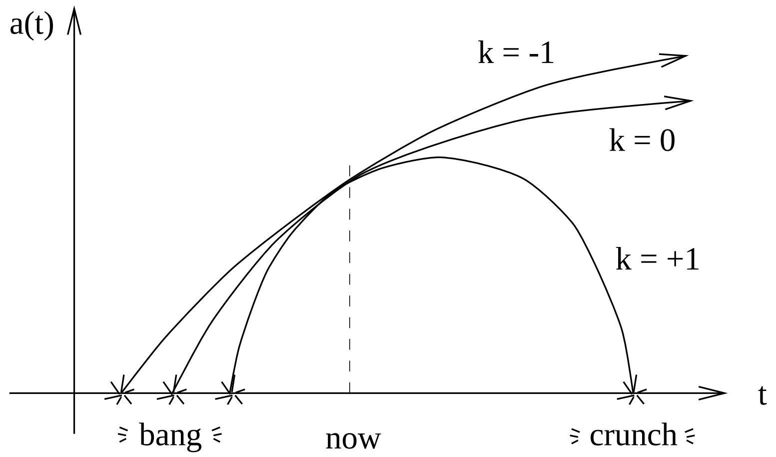
For the closed universes ($k = +1$), (8.36) becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1 \ . \tag{8.46}$$

The argument that $\rho a^2 \to 0$ as $a \to \infty$ still applies; but in that case (8.46) would become negative, which can't happen. Therefore the universe does not expand indefinitely; a possesses an upper bound $a_{\max}$. As a approaches $a_{\max}$, (8.35) implies

$$\ddot{a} \to -\frac{4\pi G}{3}(\rho + 3p)a_{\max} < 0 \ . \tag{8.47}$$

Thus $\ddot{a}$ is finite and negative at this point, so a reaches $a_{\max}$ and starts decreasing, whereupon (since $\ddot{a} < 0$) it will inevitably continue to contract to zero — the Big Crunch. Thus, the closed universes (again, under our assumptions of positive ρ and nonnegative p) are closed in time as well as space.



We will now list some of the exact solutions corresponding to only one type of energy density. For dust-only universes ($p = 0$), it is convenient to define a **development angle** $\phi(t)$, rather than using t as a parameter directly. The solutions are then, for open universes,

$$\begin{cases} a = \frac{C}{2}(\cosh\phi - 1) \\ t = \frac{C}{2}(\sinh\phi - \phi) \end{cases} \qquad (k = -1) \ , \tag{8.48}$$

for flat universes,

$$a = \left(\frac{9C}{4}\right)^{1/3} t^{2/3} \qquad (k = 0) \ , \tag{8.49}$$

and for closed universes,

$$\begin{cases} a = \frac{C}{2}(1 - \cos\phi) \\ t = \frac{C}{2}(\phi - \sin\phi) \end{cases} \qquad (k = +1) \ , \tag{8.50}$$

where we have defined

$$C = \frac{8\pi G}{3}\rho a^3 = \text{ constant} . \tag{8.51}$$

For universes filled with nothing but radiation, $p = \frac{1}{3}\rho$, we have once again open universes,

$$a = \sqrt{C'}\left[\left(1 + \frac{t}{\sqrt{C'}}\right)^2 - 1\right]^{1/2} \qquad (k = -1) , \tag{8.52}$$

flat universes,

$$a = (4C')^{1/4}t^{1/2} \qquad (k = 0) , \tag{8.53}$$

and closed universes,

$$a = \sqrt{C'}\left[1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2\right]^{1/2} \qquad (k = +1) , \tag{8.54}$$

where this time we defined

$$C' = \frac{8\pi G}{3}\rho a^4 = \text{ constant} . \tag{8.55}$$

You can check for yourselves that these exact solutions have the properties we argued would hold in general.

For universes which are empty save for the cosmological constant, either ρ or p will be negative, in violation of the assumptions we used earlier to derive the general behavior of $a(t)$. In this case the connection between open/closed and expands forever/recollapses is lost. We begin by considering $\Lambda < 0$. In this case Ω is negative, and from (8.41) this can only happen if $k = -1$. The solution in this case is

$$a = \sqrt{\frac{-3}{\Lambda}}\sin\left(\sqrt{\frac{-\Lambda}{3}}\,t\right) . \tag{8.56}$$

There is also an open ($k = -1$) solution for $\Lambda > 0$, given by

$$a = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) . \tag{8.57}$$

A flat vacuum-dominated universe must have $\Lambda > 0$, and the solution is

$$a \propto \exp\left(\pm\sqrt{\frac{\Lambda}{3}}\,t\right) , \tag{8.58}$$

while the closed universe must also have $\Lambda > 0$, and satisfies

$$a = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) . \tag{8.59}$$

These solutions are a little misleading. In fact the three solutions for $\Lambda > 0$ — (8.57), (8.58), and (8.59) — all represent the same spacetime, just in different coordinates. This spacetime, known as **de Sitter space**, is actually maximally symmetric as a spacetime. (See Hawking and Ellis for details.) The $\Lambda < 0$ solution (8.56) is also maximally symmetric, and is known as **anti-de Sitter space**.

It is clear that we would like to observationally determine a number of quantities to decide which of the FRW models corresponds to our universe. Obviously we would like to determine H_0, since that is related to the age of the universe. (For a purely matter-dominated, $k = 0$ universe, (8.49) implies that the age is $2/(3H_0)$. Other possibilities would predict similar relations.) We would also like to know Ω, which determines k through (8.41). Given the definition (8.39) of Ω, this means we want to know both H_0 and ρ_0. Unfortunately both quantities are hard to measure accurately, especially ρ. But notice that the deceleration parameter q can be related to Ω using (8.35):

$$\begin{aligned} q &= -\frac{a\ddot{a}}{\dot{a}^2} \\ &= -H^{-2}\frac{\ddot{a}}{a} \\ &= \frac{4\pi G}{3H^2}(\rho + 3p) \\ &= \frac{4\pi G}{3H^2}\rho(1+3w) \\ &= \frac{1+3w}{2}\Omega\ . \end{aligned} \tag{8.60}$$

Therefore, if we think we know what w is (*i.e.*, what kind of stuff the universe is made of), we can determine Ω by measuring q. (Unfortunately we are not completely confident that we know w, and q is itself hard to measure. But people are trying.)

To understand how these quantities might conceivably be measured, let's consider geodesic motion in an FRW universe. There are a number of spacelike Killing vectors, but no timelike Killing vector to give us a notion of conserved energy. There is, however, a Killing tensor. If $U^\mu = (1, 0, 0, 0)$ is the four-velocity of comoving observers, then the tensor

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \tag{8.61}$$

satisfies $\nabla_{(\sigma}K_{\mu\nu)} = 0$ (as you can check), and is therefore a Killing tensor. This means that if a particle has four-velocity $V^\mu = dx^\mu/d\lambda$, the quantity

$$K^2 = K_{\mu\nu}V^\mu V^\nu = a^2[V_\mu V^\mu + (U_\mu V^\mu)^2] \tag{8.62}$$

will be a constant along geodesics. Let's think about this, first for massive particles. Then we will have $V_\mu V^\mu = -1$, or

$$(V^0)^2 = 1 + |\vec{V}|^2\ , \tag{8.63}$$

where $|\vec{V}|^2 = g_{ij}V^iV^j$. So (8.61) implies

$$|\vec{V}| = \frac{K}{a} \ . \tag{8.64}$$

The particle therefore "slows down" with respect to the comoving coordinates as the universe expands. In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the universe expands.

A similar thing happens to null geodesics. In this case $V_\mu V^\mu = 0$, and (8.62) implies

$$U_\mu V^\mu = \frac{K}{a} \ . \tag{8.65}$$

But the frequency of the photon as measured by a comoving observer is $\omega = -U_\mu V^\mu$. The frequency of the photon emitted with frequency ω_1 will therefore be observed with a lower frequency ω_0 as the universe expands:

$$\frac{\omega_0}{\omega_1} = \frac{a_1}{a_0} \ . \tag{8.66}$$

Cosmologists like to speak of this in terms of the **redshift** z between the two events, defined by the fractional change in wavelength:

$$\begin{aligned} z &= \frac{\lambda_0 - \lambda_1}{\lambda_1} \\ &= \frac{a_0}{a_1} - 1 \ . \end{aligned} \tag{8.67}$$

Notice that this redshift is not the same as the conventional Doppler effect; it is the expansion of space, not the relative velocities of the observer and emitter, which leads to the redshift.

The redshift is something we can measure; we know the rest-frame wavelengths of various spectral lines in the radiation from distant galaxies, so we can tell how much their wavelengths have changed along the path from time t_1 when they were emitted to time t_0 when they were observed. We therefore know the ratio of the scale factors at these two times. But we don't know the times themselves; the photons are not clever enough to tell us how much coordinate time has elapsed on their journey. We have to work harder to extract this information.

Roughly speaking, since a photon moves at the speed of light its travel time should simply be its distance. But what is the "distance" of a far away galaxy in an expanding universe? The comoving distance is not especially useful, since it is not measurable, and furthermore because the galaxies need not be comoving in general. Instead we can define the **luminosity distance** as

$$d_L^2 = \frac{L}{4\pi F} \ , \tag{8.68}$$

where L is the absolute luminosity of the source and F is the flux measured by the observer (the energy per unit time per unit area of some detector). The definition comes from the fact that in flat space, for a source at distance d the flux over the luminosity is just one over the area of a sphere centered around the source, $F/L = 1/A(d) = 1/4\pi d^2$. In an FRW universe, however, the flux will be diluted. Conservation of photons tells us that the total number of photons emitted by the source will eventually pass through a sphere at comoving distance r from the emitter. Such a sphere is at a physical distance $d = a_0 r$, where a_0 is the scale factor when the photons are observed. But the flux is diluted by two additional effects: the individual photons redshift by a factor $(1+z)$, and the photons hit the sphere less frequently, since two photons emitted a time δt apart will be measured at a time $(1+z)\delta t$ apart. Therefore we will have

$$\frac{F}{L} = \frac{1}{4\pi a_0^2 r^2 (1+z)^2} , \tag{8.69}$$

or

$$d_L = a_0 r(1+z) . \tag{8.70}$$

The luminosity distance d_L is something we might hope to measure, since there are some astrophysical sources whose absolute luminosities are known ("standard candles"). But r is not observable, so we have to remove that from our equation. On a null geodesic (chosen to be radial for convenience) we have

$$0 = ds^2 = -\mathrm{d}t^2 + \frac{a^2}{1-kr^2}\mathrm{d}r^2 , \tag{8.71}$$

or

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr}{(1-kr^2)^{1/2}} . \tag{8.72}$$

For galaxies not too far away, we can expand the scale factor in a Taylor series about its present value:

$$a(t_1) = a_0 + (\dot{a})_0(t_1 - t_0) + \frac{1}{2}(\ddot{a})_0(t_1 - t_0)^2 + \ldots . \tag{8.73}$$

We can then expand both sides of (8.72) to find

$$r = a_0^{-1}\left[(t_0 - t_1) + \frac{1}{2}H_0(t_0 - t_1)^2 + \ldots\right] . \tag{8.74}$$

Now remembering (8.67), the expansion (8.73) is the same as

$$\frac{1}{1+z} = 1 + H_0(t_1 - t_0) - \frac{1}{2}q_0 H_0^2 (t_1 - t_0)^2 + \ldots . \tag{8.75}$$

For small $H_0(t_1 - t_0)$ this can be inverted to yield

$$t_0 - t_1 = H_0^{-1}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right] . \tag{8.76}$$

Substituting this back again into (8.74) gives

$$r = \frac{1}{a_0 H_0}\left[z - \frac{1}{2}\left(1 + q_0\right) z^2 + \ldots\right] . \tag{8.77}$$

Finally, using this in (8.70) yields **Hubble's Law**:

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0)z^2 + \ldots\right] . \tag{8.78}$$

Therefore, measurement of the luminosity distances and redshifts of a sufficient number of galaxies allows us to determine H_0 and q_0, and therefore takes us a long way to deciding what kind of FRW universe we live in. The observations themselves are extremely difficult, and the values of these parameters in the real world are still hotly contested. Over the next decade or so a variety of new strategies and more precise application of old strategies could very well answer these questions once and for all.