which there is a value of ϕ which enters into the system of body-equations by way of the boundary-conditions will be called a *boundary-point.*

The values of any function of position at these two classes of points will be distinguished as *body values* and *boundary values*, or synonymously as *body-numbers* and *boundary-numbers.*

Problems are divided into two main classes according as the integral can or cannot be stepped out from a part of the boundary. They are discussed in § 2 and § 3 respectively.

§ 1·2. *Errors due to Finite Differences.*—Having solved an equation using the simple expressions of § 1·1 for the differential coefficients, it remains to enquire how much in error the integral may be. A rule of apparently universal application is to take smaller co-ordinate differences and repeat the integration; and, if necessary, extrapolate in the manner explained below.

It is known* that when central differences are used, the expansions of the differential coefficients of a function in terms of its differences contain only alternate powers of the co-ordinate difference h. The same is true for partial differential coefficients and for products of differential coefficients. Consequently the error of the representation of any differential expression by central differences is of the form $h^2F_2(x, y, z) + h^4F_4(x, y, z) +$ terms in higher powers of h^2, where F_2, F_4, &c., are independent of h.†

Next, as to the error of the finite-difference-integral ϕ. This is the infinitesimal integral of a differential equation having the error $h^2F_2(x, y, z)+h^4F_4(x, y, z)+$, &c. Let $\wp$ be the integral of the correct differential equation. Then, if we write

$$\phi(x, y, z) = \wp(x, y, z)+m\psi_1(x, y, z)+m^2\psi_2(x, y, z)+ \text{ terms}$$

in higher powers of m, it follows that a differential expression of any order and degree for ϕ differs from the corresponding one for $\wp$ by

$$m \times (\text{a function of the differential coefficients of } \wp \text{ and of } \psi_1) + \text{ terms in } m^2,\ m^3, \text{ \&c.},$$

provided only that m is independent of the co-ordinates. Now, identifying m with h^2, it follows that: *the errors of the integral and of any differential expressions derived from it, due to using the simple central differences of* § 1·1 *instead of differential coefficients, are of the form*

$$h^2f_2(x, y, z)+h^4f_4(x, y, z)+h^6f_6(x, y, z)+, \text{ \&c.}$$

Consequently, if the equation be integrated for several different values of h, extrapolation on the supposition that the error is of this form will give numbers very close to the infinitesimal integral. When h is small enough the error is simply

* W. F. SHEPPARD, "Central-Difference Formulæ," 'Proc. Lond. Math. Soc.,' vol. xxxi. (1899).

[† *Note added January* 21, 1910.—It is assumed that the co-ordinate axes in the tables which are compared are parallel, for the error at a fixed point and for a fixed value of h may depend on the direction of the axes.]

proportional to h^2. Peculiarities present themselves on the boundary, but it is easy to see that errors will be of the form $h^2f_2+h^4f_4+$, &c., provided that in passing from one table to another each part is either kept infinitesimally correct, or else is worked by differences whose size in the one table bears a constant ratio to that in the other.

An extrapolation can only be made where the tabular points of the several tables coincide with one another. It is conceivable that in the future some method will be found of defining a continuous function in terms of the discrete body and boundary values, so that this continuous function shall have an error of the form $h^2f_2(x, y, z)+h^4f_4(x, y, z)+$, &c., everywhere. Extrapolation would then be possible everywhere.

An excellent illustration is afforded by Lord RAYLEIGH's account of the vibration of a stretched string of beads ('Sound,' vol. I., § 121). He gives the frequency of the fundamental for the same mass per unit length concentrated in various numbers of beads. This is reproduced below in the table. The co-ordinate difference h is inversely as one plus the number of beads, not counting beads at the fixed ends.

Number of free beads + one . .	2	3	4	5	10	20	40	∞
Ratio of frequency to that of continuous string	·9003	·9549	·9745	·9836	·9959	·9990	·9997	unity
Error in representation of continuous string by string of beads	·0997	·0451	·0255	·0164	·0041	·0010	·0003	·0000
Ratio of error to square of co-ordinate difference × a constant	·3988	·4059	·4080	·4091	·4107	·4111	·4112	·4112

The degree of constancy of the last line shows that if we found the frequency for one bead and for three, then extrapolation, on the assumption that the error is proportional to h^2, would give us the frequency for the continuous string to about one part in 1000; which is as near as we could get by twenty beads and no extrapolation. While extrapolation from the exact solutions for four beads and for nine would leave an error of only one in 50,000. Other examples of extrapolation will be found in § 3·1.

§ 2. *Procedure when the Conditions allow the Integral to be Marched out from a Part of the Boundary.*

§ 2·0. *Historical.*—Step-by-step arithmetical methods of solving ordinary difference equations have long been employed for the calculation of interest and annuities. Recently their application to differential equations has been very greatly improved by the introduction of rules allied to those for approximate quadrature. The papers referred to are:—

RUNGE, "Über die numerische Auflösung von Differentialgleichungen," 'Math. Ann.,' Bd. 46. Leipzig, 1895.

Again, from the size and distribution of $\mathfrak{D}'\phi$ combined with a knowledge of various integrals of $\mathfrak{D}'\phi = f(x, y)$, a rough estimate of the errors in ϕ can frequently be made. Again for certain equations* the method of contour integration applied to a circle affords a check on the value of ϕ at its centre. This method is very rapid, and it is particularly advantageous when applied to a circle enclosing many body values, for then a repetition of the approximation process would be correspondingly tedious.

§ 3·2·5. *Routine of Approximation.—Time and Cost.*—To anyone setting out on a problem I offer the following experience as a guide in forming estimates :—It was found convenient to enter certain stages on a table with large squares, each divided into compartments. Thus for $\frac{\delta^4\phi}{\partial x^4} + 2\frac{\delta^4\phi}{\delta x^2 \delta y^2} + \frac{\delta^4\phi}{\delta y^4} = 0$, one of the squares is shown in the annexed table. All the quantities in it refer to the central point of the square. The intermediate stages are done on rough paper and thrown away. So far I have paid piece rates for the operation $\delta_x^2 + \delta_y^2$ of about $\frac{n}{18}$ pence per co-ordinate point, n being the number of digits. The chief trouble to the computers has been the intermixture of plus and minus signs. As to the rate of working, one of the quickest boys averaged 2,000 operations $\delta_x^2 + \delta_y^2$ per week, for numbers of three digits, those done wrong being discounted.

TABLE.

ϕ_1	$\nabla^2\phi_1$	$\nabla^4\phi_1$
ϕ_2	$\nabla^2\phi_2$	$\nabla^4\phi_2$
ϕ_3	&c.	

§ 3·3. *Relative Merits of Simultaneous Equations and of Successive Approximation.*—The method of simultaneous equations may be applied to differential equations of any order and degree. It gives results which are exact for finite differences. It is necessary in discussions as to the existence and properties of the integrals of difference equations. But for actually calculating the integrals the labour becomes very great as the number of unknowns increases, and is of a sort which a clerk will not easily do. Large numbers of digits have to be dealt with, and a single mistake generally throws the result altogether out.

The successive approximation methods of § 3·2 have only been applied to a limited class of linear equations. The results are not exact even for finite differences. But the bulk of the work can be done by clerks who need not understand algebra or calculus. Small and infrequent mistakes, or taking only a small number of digits, do not prevent one arriving at a fairly correct result. Nevertheless, it has been found best to have everything worked in duplicate.

The method of successive approximation to the surface $z = \phi_u = f(x, y)$ reminds one of the manufacture of plane metallic surfaces. The initial form of the surface is arbitrary in both cases. The essential things in both cases are a method of testing the work at any stage, a tool with which to alter the surface and judgment in using it. Methods of testing the arithmetic have been described in § 3·2·4 above. Our

* These include $\nabla^2\phi = 0$, $\nabla_1{}^4\phi = 0$. See a paper by BOGGIO, 'Jahrb. Fortschritte Math.,' 1900, p. 740.