PHYS 170 Section 101

Lecture 33

November 28, 2018

Nov 28—Announcements

- Slides from yesterday's final exam review sessions are posted on Canvas

Lecture Outline/Learning Goals

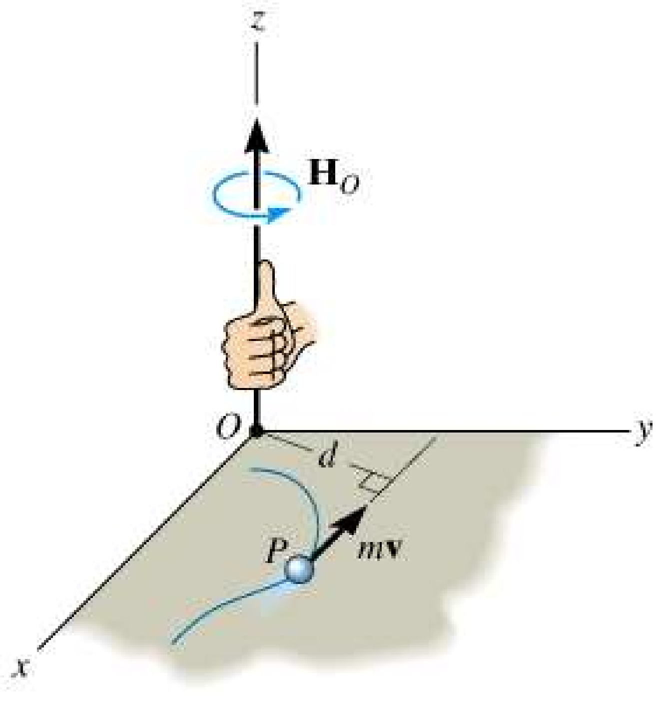
- 15.5 Angular Momentum
- 15.6 Relation Between Moment of a Force and Angular Momentum
- 15.7 Principle of Angular Impulse and Momentum (including Conservation of Angular Momentum)
- Worked example using conservation of angular momentum

15.5 Angular Momentum

- The **angular momentum** of a particle about a point *O* can be viewed as the "moment" of the particle's linear momentum about *O*

- Computation of angular momentum from linear momentum is analogous to the computation of moment from force, so angular momentum, $\mathbf{H}_O$, is sometimes called the **moment of momentum**

- **Important:** Angular momentum is another **vector** quantity, and again, like the moment of a force, it is always defined with respect to some specific point (the point must be fixed during the analysis of the physics)



Angular momentum: Scalar Formulation

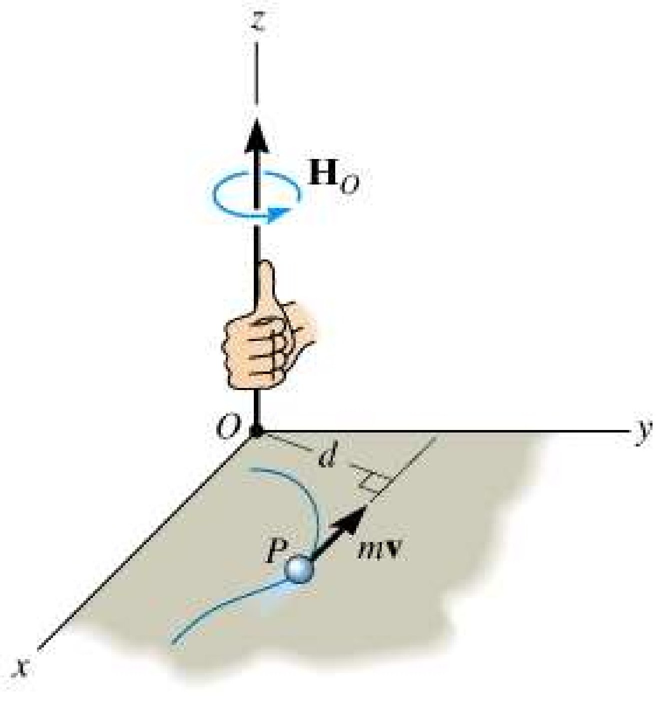
- Consider a particle *P*, moving on a curve in the *x*-*y* plane, with linear momentum $m\mathbf{v}$, as shown in the figure

- Then the **magnitude** of the particle's angular momentum about *O* is given by

$$(H_O)_z = (d)(mv)$$

where *d* is the moment arm (perpendicular distance) from *O* to the line of action of $m\mathbf{v}$

- The **direction** of the particle's angular momentum is perpendicular to the *x*-*y* plane (i.e. along the *z*-axis), with a sense given by the right hand rule, where the fingers of the right hand curl in the direction of the rotation of $m\mathbf{v}$ about the *z*-axis



Angular momentum: Scalar Formulation

- As is the case for linear momentum, there are no special units for angular momentum

- From the previous expression

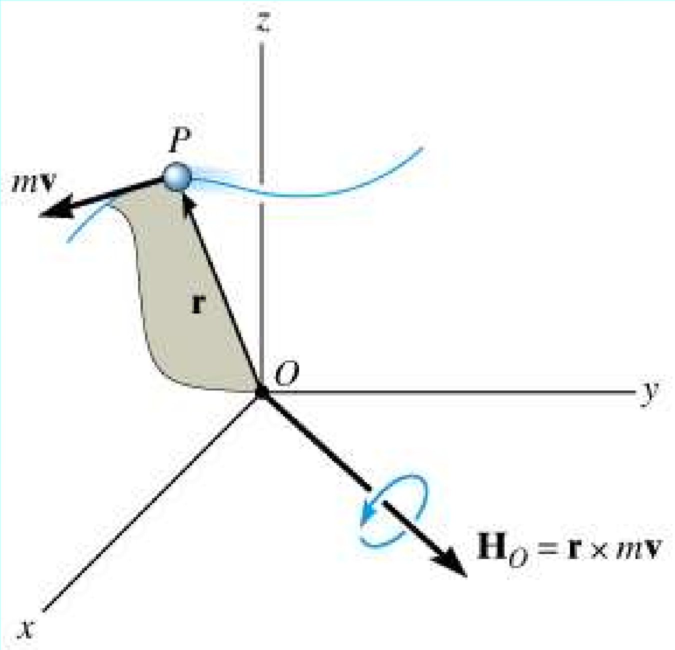
$$(H_O)_z = dmv$$

we can deduce the following composite units

- **Units for Angular Momentum**

  - **SI:** $\mathrm{kg \cdot m^2/s}$

  - **FPS:** $\mathrm{slug \cdot ft^2/s}$



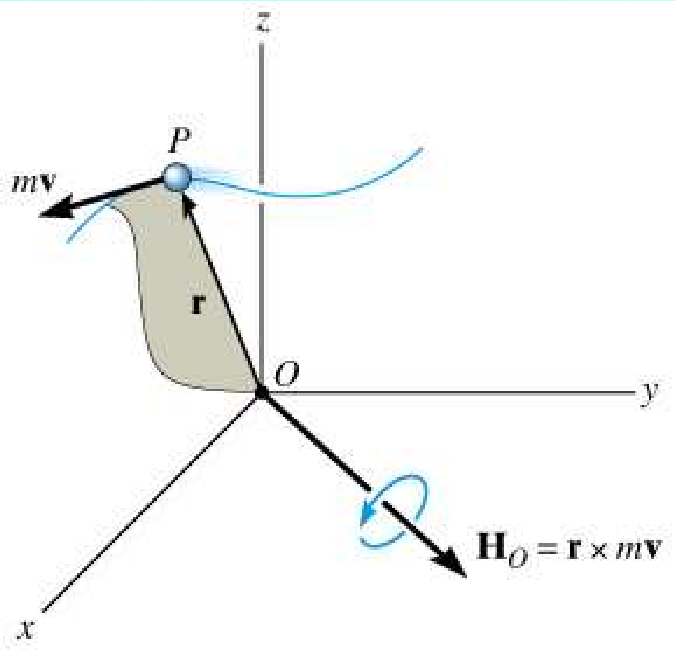
Angular momentum: Vector Formulation

- Now consider a particle, P, moving along a general 3D path, with position vector, $\mathbf{r}$, relative to the point O with respect to which the angular momentum is to be calculated, and with momentum $m\mathbf{v}$ as shown in the figure

- Then the (vector) angular momentum $\mathbf{H}_O$, of the particle about O is defined in terms of a cross product as follows:

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

- Note that, as follows from the basic property of all cross products, the direction of $\mathbf{H}_O$ is perpendicular to both $\mathbf{r}$ and $m\mathbf{v}$, and is given by the usual right hand rule for cross products



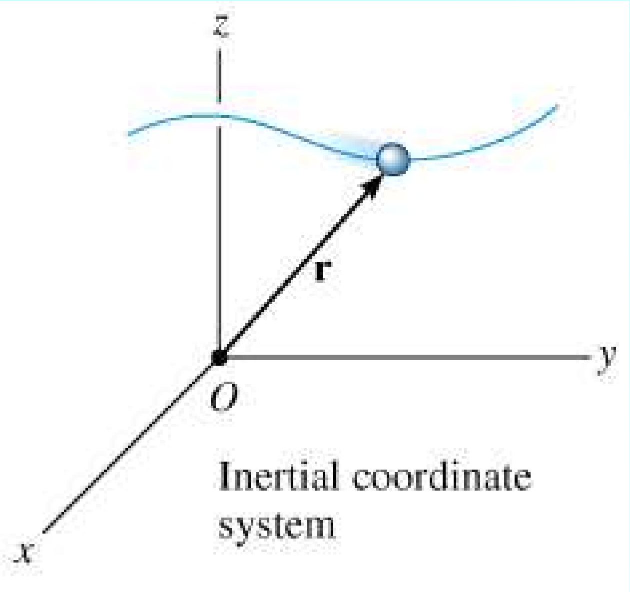
Angular momentum: Vector Formulation

- As is the case when computing the moment of a force, provided that the Cartesian components of **r** and $m\mathbf{v}$ can be determined, then the angular momentum can be conveniently evaluated using the following determinant

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

- **NOTE:** For most (if not all) of the problems involving angular momentum that we will consider, the geometry of the situation is sufficiently simple that the scalar approach will suffice

15.6 Relation Between Moment of a Force and Angular Momentum



- Consider a particle moving along some 3D trajectory as shown in the figure, such that at any given time it has a position vector **r** with respect to the origin *O* of an inertial coordinate system

- Assuming that the particle mass is constant, Newton's second law is

$$\Sigma\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt}$$

- Note that this equation relates the resultant force acting on the particle to the time rate of change of the particle's linear momentum

- Now consider taking the cross product of **r** with both sides of this equation

$$\mathbf{r}\times\Sigma\mathbf{F} = \mathbf{r}\times\left(m\frac{d\mathbf{v}}{dt}\right)$$

Relation Between Moment of a Force and Angular Momentum (continued)

- We recognize the left hand side of the above equation as the sum of all the moments about O due to the forces acting on the particle

$$\Sigma\mathbf{M}_O = \mathbf{r}\times\Sigma\mathbf{F}$$

- We can now show that this moment sum is equal to the time rate of change of the particle's angular momentum

- Using the vector definition of angular momentum and a formula from Appendix C of the text for the derivative of a cross product, we have

$$\begin{aligned}\frac{d\mathbf{H}_O}{dt} &= \frac{d}{dt}\big(\mathbf{r}\times(m\mathbf{v})\big) = \frac{d\mathbf{r}}{dt}\times(m\mathbf{v}) + \mathbf{r}\times\left(m\frac{d\mathbf{v}}{dt}\right)\\ &= m(\mathbf{v}\times\mathbf{v}) + \mathbf{r}\times\left(m\frac{d\mathbf{v}}{dt}\right) \qquad (\mathbf{v}\times\mathbf{v}=\mathbf{0}!)\\ &= \mathbf{r}\times\left(m\frac{d\mathbf{v}}{dt}\right)\end{aligned}$$

Relation Between Moment of a Force and Angular Momentum (continued)

- Recalling that crossing **r** with both sides of Newton's second law for a particle gave us

$$\mathbf{r} \times \Sigma \mathbf{F} = \mathbf{r} \times \left(m \frac{d\mathbf{v}}{dt} \right)$$

we see that we have

$$\boxed{\Sigma \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}}$$

- In words: The resultant moment about a point *O* of all forces acting on a particle is equal to the time rate of change of the angular momentum of the particle, also computed about *O*

15.7 Angular Impulse and Momentum Principles

Principle of Angular Impulse and Momentum (single particle)

- For a single particle we have

$$\Sigma \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}$$

- Paralleling the development of the principle of linear impulse and momentum, we will multiply both sides of this equation by *dt*, then integrate over a time interval $t_1 \leq t \leq t_2$, with initial and final angular momenta given by

$$\mathbf{H}_O(t_1) = (\mathbf{H}_O)_1 = \mathbf{r} \times m\mathbf{v}_1$$
$$\mathbf{H}_O(t_2) = (\mathbf{H}_O)_2 = \mathbf{r} \times m\mathbf{v}_2$$

- Performing the integration, we get our first form of the **principle of angular impulse and momentum**

$$\Sigma \int_{t_1}^{t_2} \mathbf{M}_O \, dt = \int_{(\mathbf{H}_0)_1}^{(\mathbf{H}_0)_2} d\mathbf{H}_O = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

Principle of Angular Impulse and Momentum (single particle) [continued]

- Again, analogously to the case of linear impulse, we call the time integral of a moment an **angular impulse** – i.e. given a fixed point O in an inertial reference frame, and a moment $\mathbf{M}_O$, about that point

$$\text{angular impulse} = \int_{t_1}^{t_2} \mathbf{M}_O \, dt$$

- **Units for Angular Impulse**
  - **SI:** $\mathrm{N \cdot m \cdot s}$
  - **FPS:** $\mathrm{lb \cdot ft \cdot s}$

Principle of Angular Impulse and Momentum (single particle) [continued]

- In the case that $\mathbf{M}_O$ is due to a force, $\mathbf{F}$, we have

$$\text{angular impulse} = \int_{t_1}^{t_2} \mathbf{M}_O dt = \int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F})\, dt$$

  where, as usual, $\mathbf{r}$ is a position vector from O to any point along the line of action of $\mathbf{F}$

- We have previously derived the principle of angular impulse and momentum

$$\Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

  which we can write in the form most often used for problem solving

$$\boxed{(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2}$$

Conservation of Angular Momentum

- Now consider the case where the (vector) sum of the angular impulses acting on the particle during the time interval vanishes, which will most frequently happen when there are **no** angular impulses acting on the particle

- We then have

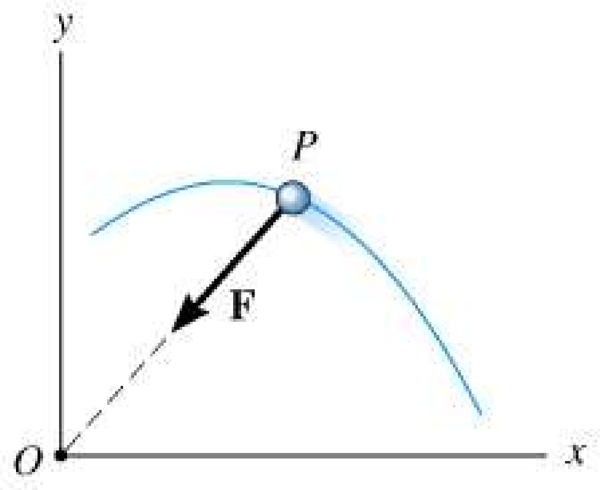
$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

which is known as the **conservation of angular momentum**

- **NOTE:** As is the case for conservation of linear momentum, because angular momentum is a **vector** quantity, depending on the nature of the angular impulses that are applied to the particle, it is possible that angular momentum may be conserved in some direction (i.e. about some specific axis, like the z-axis) but not in others

Conservation of Angular Momentum (continued)

- In addition, although it may be the case that **both** linear and angular momentum are conserved (which will happen, for example, if there are **no** external impulses acting on the particle), it is also possible for angular momentum to be conserved in cases where linear momentum is not

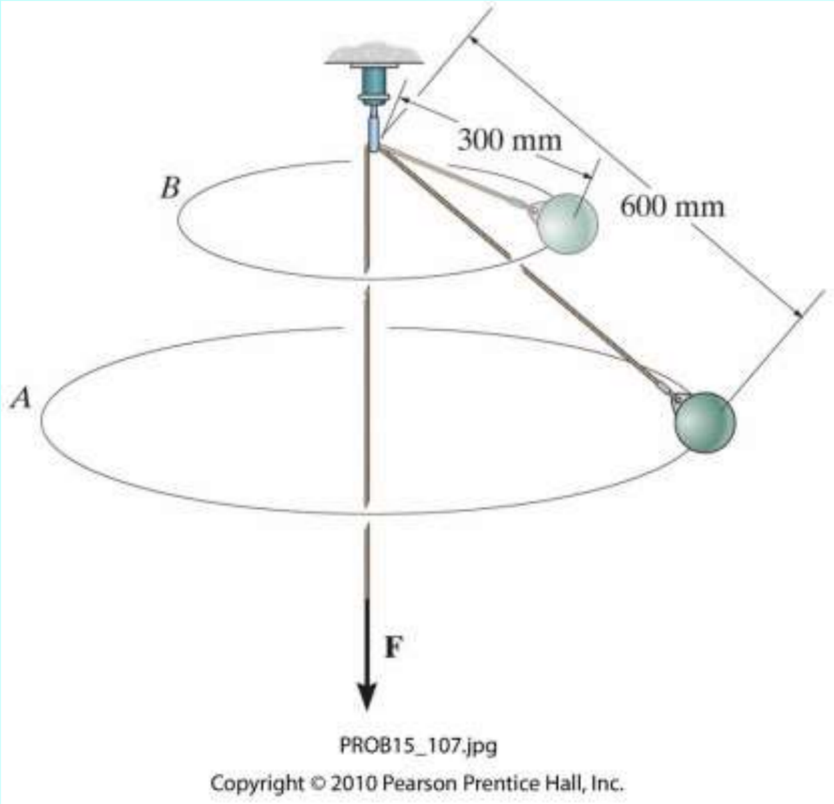


- A good example of this is the situation where the particle moves under the influence of a so-called **central force**, as shown in the figure

- Here, as the particle, *P,* moves along the path, the single force **F** which acts on it is always directed towards the origin, *O*, through which the *z* axis passes

- Since **F** creates no moment or angular impulse about *O,* or the *z* axis, the angular momentum of the particle about the *z* axis is conserved, whereas the particle’s linear momentum will generally be a changing function of time

Problem 15-108 (Page 293, 14th edition)



The 2 kg ball rotates around the horizontal circular path *A* with speed 1.5 m/s

(1) Determine the radius of the circular path

(2) Determine the tension in the cord

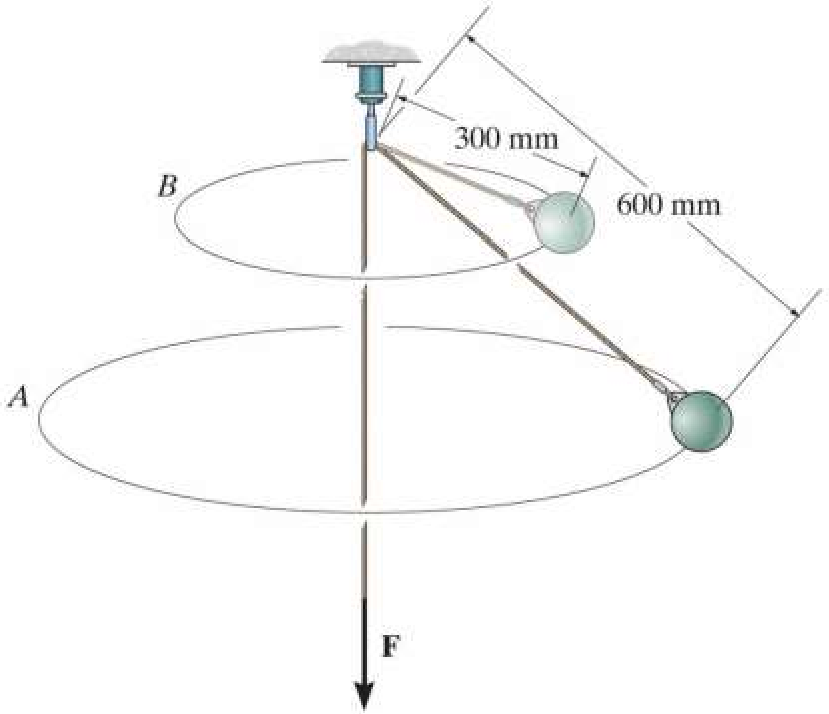
The tension force on the cord is increased. The ball rises and rotates around the horizontal circular path *B*

(3) Determine the radius of the circular path

(4) Determine the speed of the ball

(5) Determine the tension in the cord

(6) Determine the work done by the tension force in moving the ball from path *A* to path *B*



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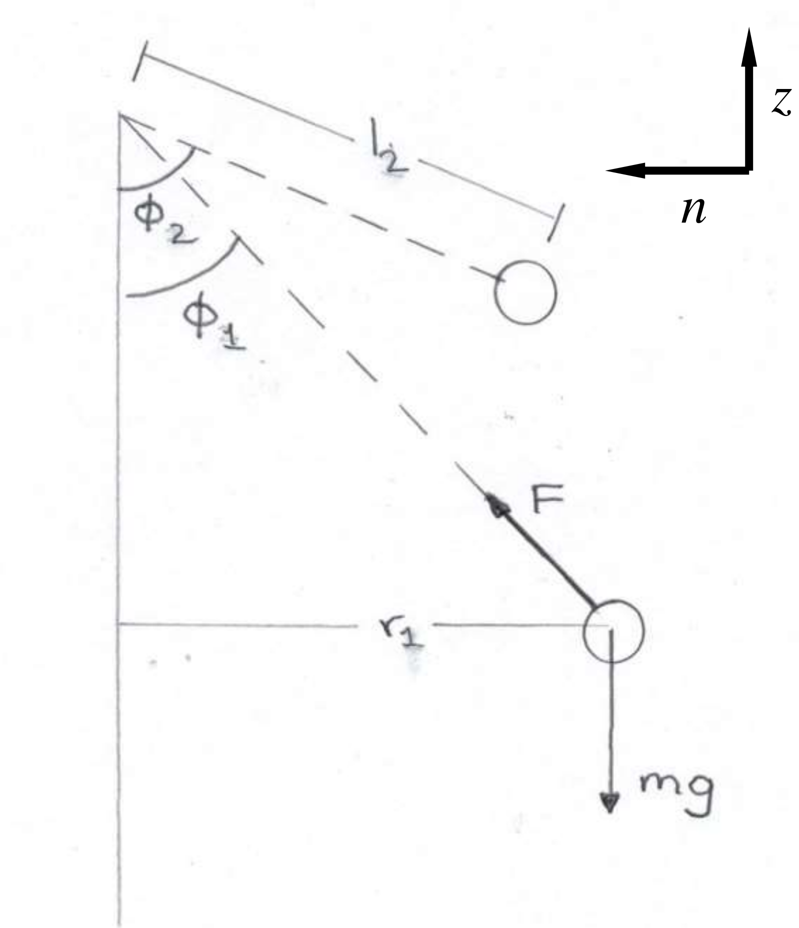

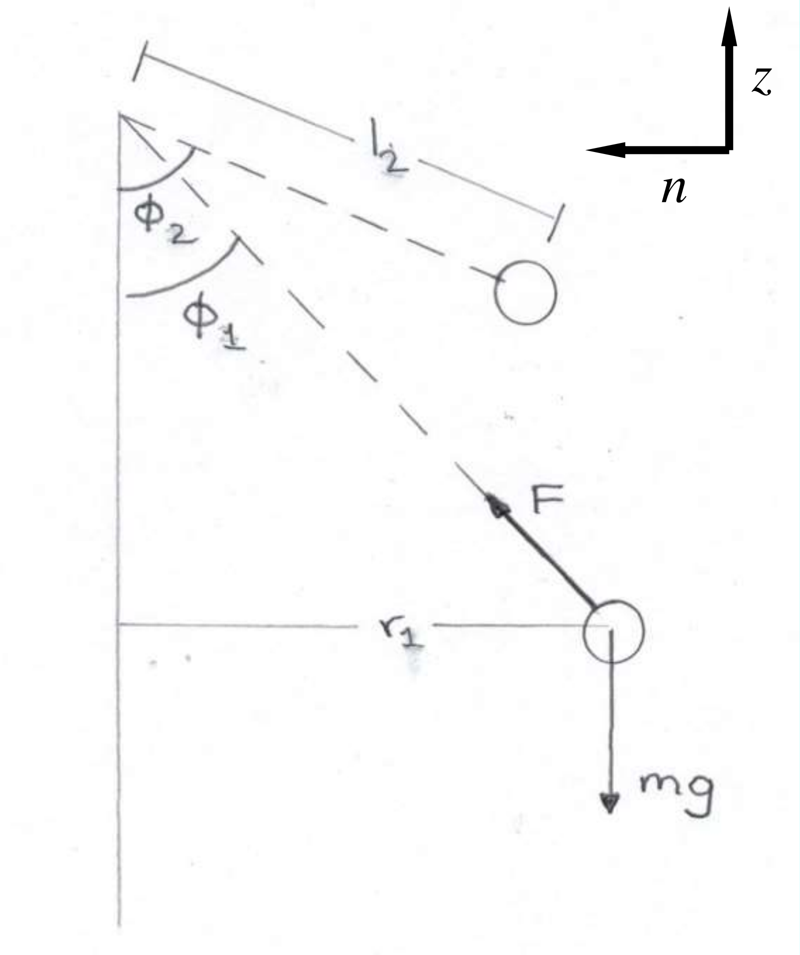

Solution strategy

Use equations of motion in nz coordinates and geometry to determine ϕ_1, then compute r_1 and F_1

Use conservation of angular momentum to determine ϕ_2, then compute r_2, v_2 and F_2

Use energy balance equation (principle of work energy) to compute work done by tension force

Data

$m = 2 \text{ kg}$

Initial circular path: $l_1 = 600 \text{ mm} = 0.6 \text{ m}$ $v_1 = 1.5 \text{ m/s}$

Final circular path: $l_2 = 300 \text{ mm} = 0.3 \text{ m}$

Motion in a circular path

$\sum F_n = ma_n :$ $$F \sin\phi = \frac{mv^2}{r}$$

$\sum F_z = ma_z :$ $$F \cos\phi - mg = 0 \quad \Rightarrow \quad F \cos\phi = mg$$

Divide first equation by second $\Rightarrow$ $\tan\phi = \dfrac{v^2}{rg}$ $\Rightarrow$ $v = \sqrt{gr\tan\phi}$

Motion in a circular path (cont.)

Radius: $r = l \sin\phi$ (1)

Speed: $v = \sqrt{gr\tan\phi}$ (2)

Force: $F = \dfrac{mg}{\cos\phi}$ (3)

Initial circular path

Using the data and Eq. (1), Eq. (2) is

$$\sqrt{9.81(0.6)\sin\phi_1 \tan\phi_1} = 1.5$$

Solving for ϕ_1 (using "solver" on calculator or Wolfram Alpha)

$$\phi_1 = 34.21^\circ$$

Then

$$r_1 = l \sin\phi_1 = (600)(\sin 34.21^\circ) = 337 \text{ mm}$$

$$F_1 = \frac{mg}{\cos\phi_1} = \frac{(2)(9.81)}{\cos 34.21^\circ} = 23.7 \text{ N}$$

Final circular path

The weight of the ball acts parallel to the z-axis and the tension acts through it. Thus, there are no moments about the z-axis and the angular momentum about it is therefore conserved.

Using the fact that the mass of the ball is constant, we can write the conservation of angular momentum about the z-axis as

$$r_2 v_2 = r_1 v_1 \qquad (4)$$

Using the data, the value of r_1 and Eq. (1), Eq. (4) is

$$0.3\sin\phi_2\sqrt{9.81(0.3)\sin\phi_2\tan\phi_2} = 0.3374(1.5)$$

Solving this equation for ϕ_2 ("solver"/Wolfram Alpha) yields

$$\phi_2 = 57.87^\circ$$

And using the same relations as previously (and $v = \sqrt{gr\tan\phi}$), we find

$r_2 = 254\text{ mm}$ $\qquad$ $v_2 = 1.99\text{ m/s}$ $\qquad$ $F_2 = 36.9\text{ N}$

Work done by the tension force

Choose as datum for the gravitational potential energy the bottom of the ring through which the cord is threaded. Then the heights of the mass from the datum are given by

$$h_1 = -l_1 \cos\phi_1 = -0.600 \cos 34.21^\circ = -0.4962 \text{ m}$$

$$h_2 = -l_2 \cos\phi_2 = -0.300 \cos 57.87^\circ = -0.1596 \text{ m}$$

Energy balance equation

Let $(U_{\text{tension force}})_{1-2}$ be the work done by the tension force $\vec{F}$ in raising the mass from circular path A to circular path B. Then

$$\frac{1}{2}mv_1^2 + mgh_1 + (U_{\text{tension force}})_{1-2} = \frac{1}{2}mv_2^2 + mgh_2$$

Thus, we have (Exercise: check calculations)

$$(U_{\text{tension force}})_{1-2} = \left[\frac{1}{2}mv_2^2 + mgh_2\right] - \left[\frac{1}{2}mv_1^2 + mgh_1\right]$$

$$= 8.31\ \text{J}$$