PHYS 170 Section 101
Lecture 26
November 7, 2018

Nov 7—Announcements

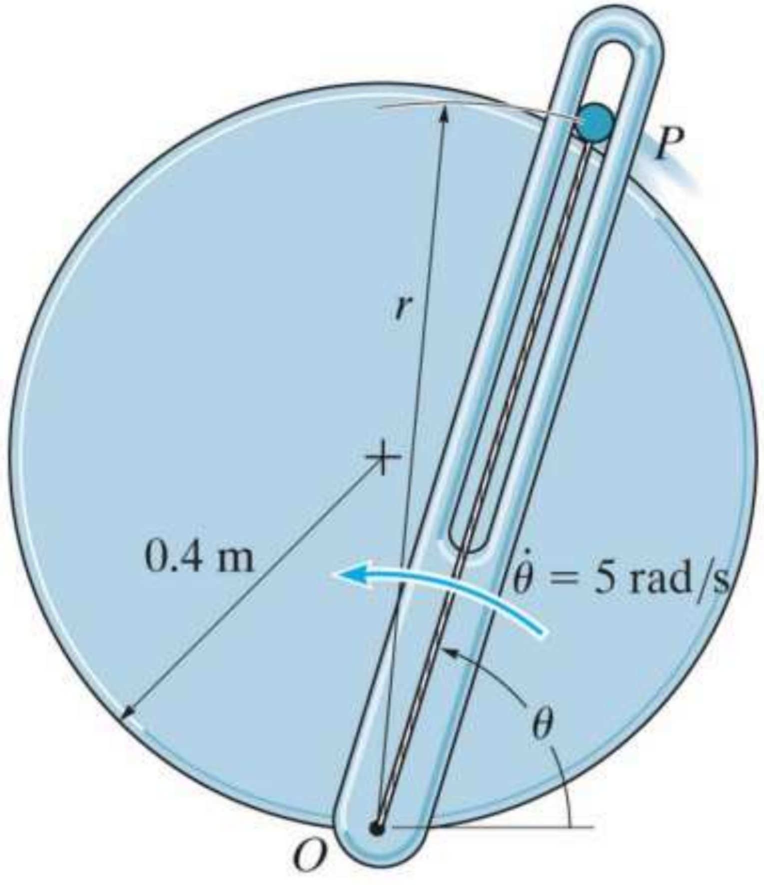
- Midterm review slides have been posted on Canvas in Lectures section

Lecture Outline/Learning Goals

- Finish particle dynamics in polar coordinates problem from last day
- Start Chapter 14: Kinetics of a Particle: Work and Energy
  - 14.1 The Work of a Force
  - 14.2   Principle of Work and Energy

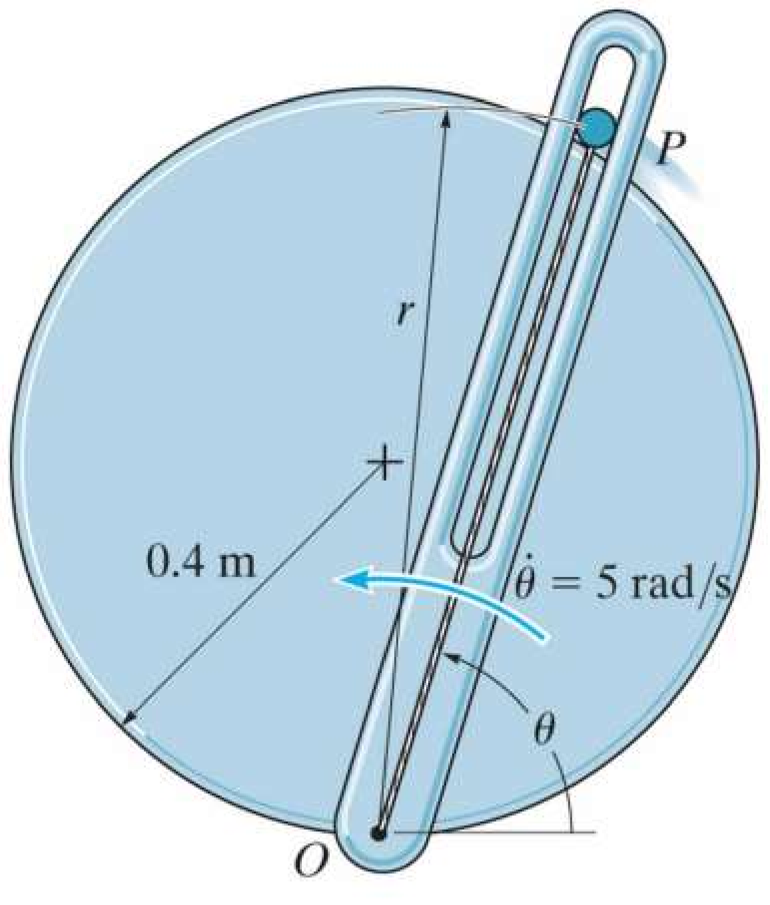
Problem 13-110 (Page 153, 13th edition)

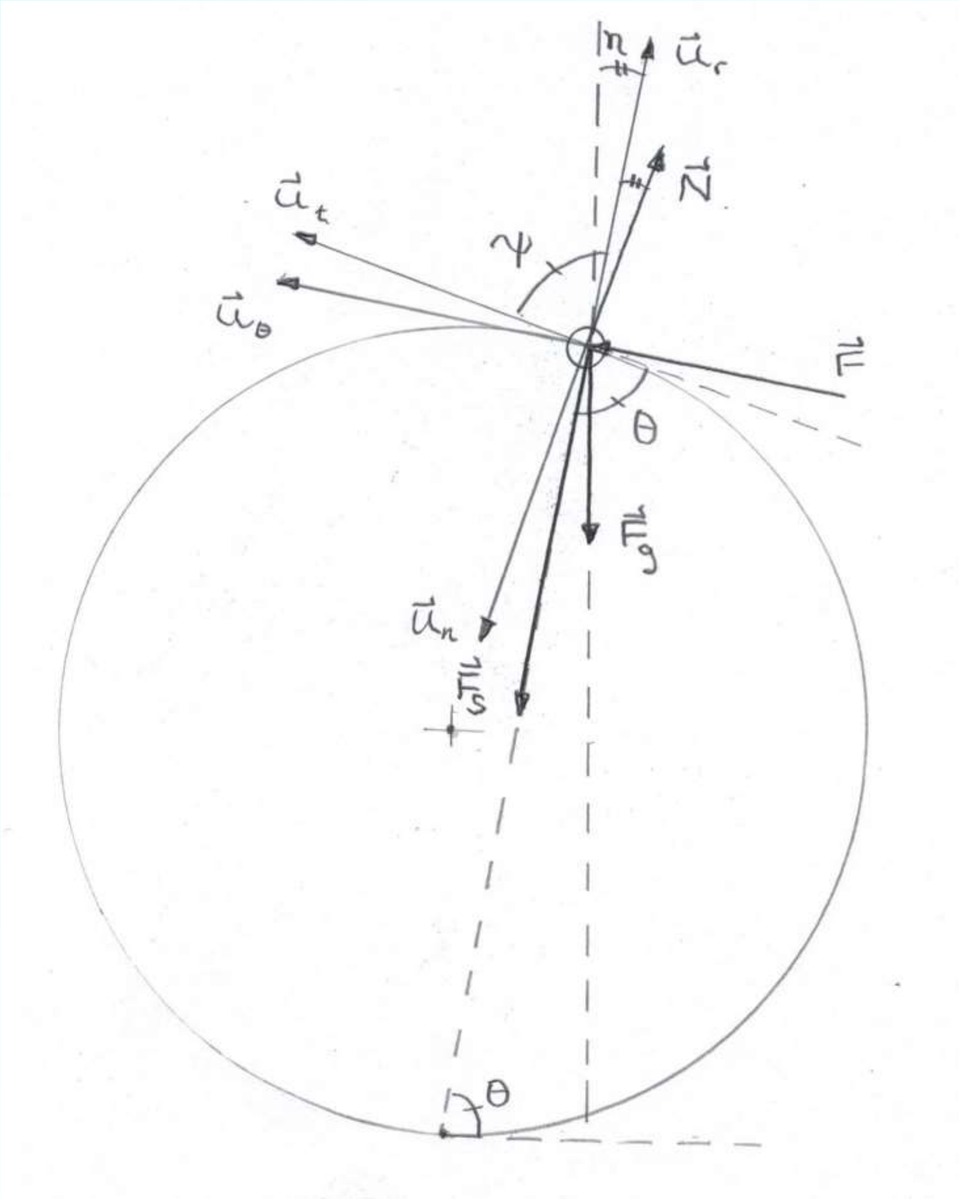
The slotted guide moves the 150 g particle P around the 0.4 m radius circular disk. Motion is in the vertical plane. Attached to P is an elastic cord extending from O. The cord has stiffness 30 N/m and unstretched length 0.25 m. Friction may be neglected.



PROB13_105-106.jpg

(1) Determine the force of the guide on P and the normal force of the disk on P when $\theta = 70^\circ$, $\dot{\theta} = 5$ rad/s and $\ddot{\theta} = 2$ rad/s^2.



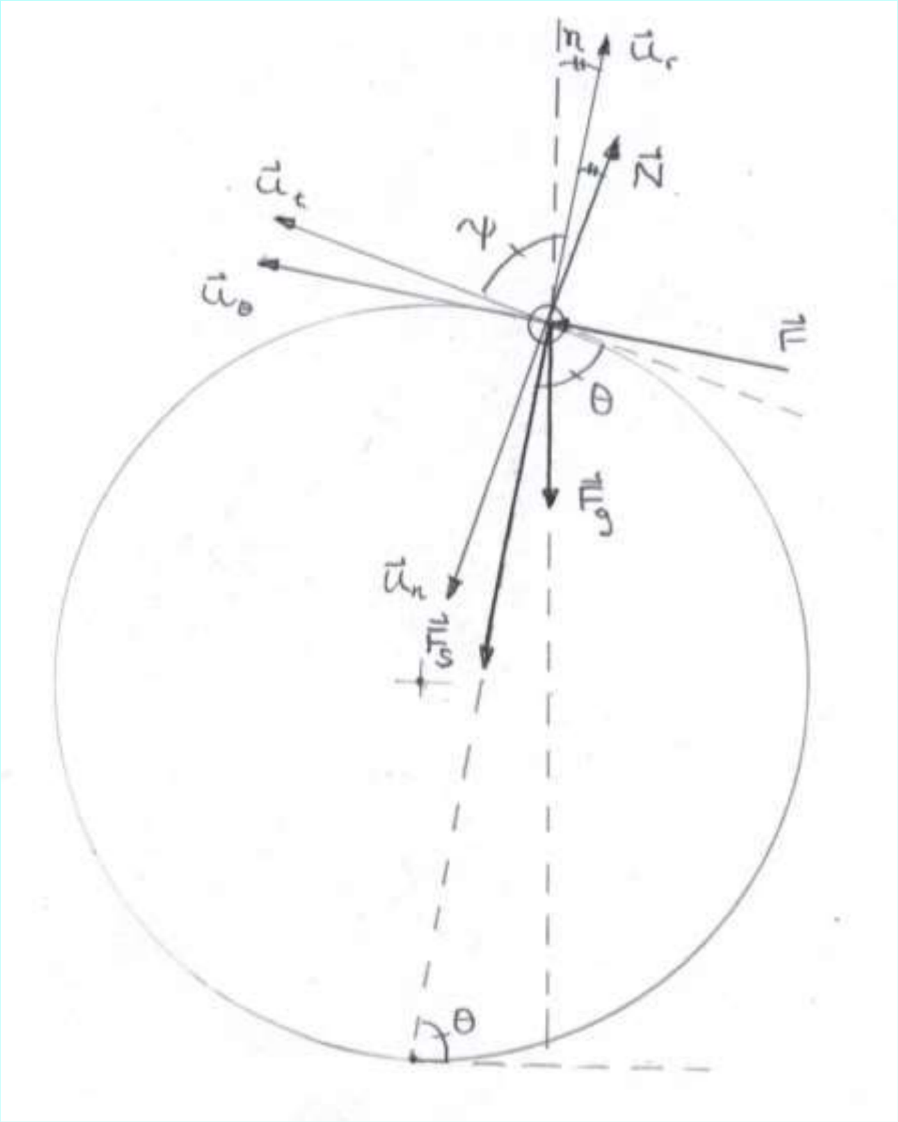


From geometry

$\psi = \theta$

$\eta = 90 - \psi = 90 - \theta$

n and y axes both make an angle η with r axis



Solution strategy

Determine acceleration components in polar coordinates, then determine requested forces from equations of motion using acceleration components and other forces (elastic cord, weight)

As before, determination of acceleration components requires computation of various derivatives ($\dot{r}$, $\ddot{r}$) and we must also determine angle, ψ, between radial and tangential unit vectors

Exercise: Refer to the original diagram. Show that $r(\theta)$ is given by $r(\theta) = 2R\sin\theta$, where R is the radius of the disk.

Data

$m = 150\text{ g}$ $\quad g = 9.81\text{ m/s}^2$ $\quad r(\theta) = 2R\sin\theta$ $\quad R = 0.4\text{ m}$

$k = 30\text{ N/m}$ $\quad r_0 = 0.25\text{ m}$

$\theta = 70^\circ$ $\quad \dot{\theta} = 5\text{ rad/s}$ $\quad \ddot{\theta} = 2\text{ rad/s}^2$

Derivatives (Exercise: verify all calculations here and below)

$$\dot{r} = 2R\dot{\theta}\cos\theta \qquad \ddot{r} = 2R\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)$$

When $\theta = 70^\circ$

$r = 0.7518\text{ m}$ $\quad \dot{r} = 1.3680\text{ m/s}$ $\quad \ddot{r} = -18.247\text{ m/s}^2$

Acceleration

$$\vec{a} = a_r \vec{u}_r + a_\theta \vec{u}_\theta$$

When $\theta = 70^\circ$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -37.040 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 15.184 \text{ m/s}^2$$

Angle between tangential and radial unit vectors

From geometry (refer to FBD): $\psi = \theta$

Alternatively

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{2R\sin\theta}{2R\cos\theta} = \tan\theta \quad \Rightarrow \quad \psi = \theta \quad \text{and} \quad \eta = 90^\circ - \theta = 20^\circ$$

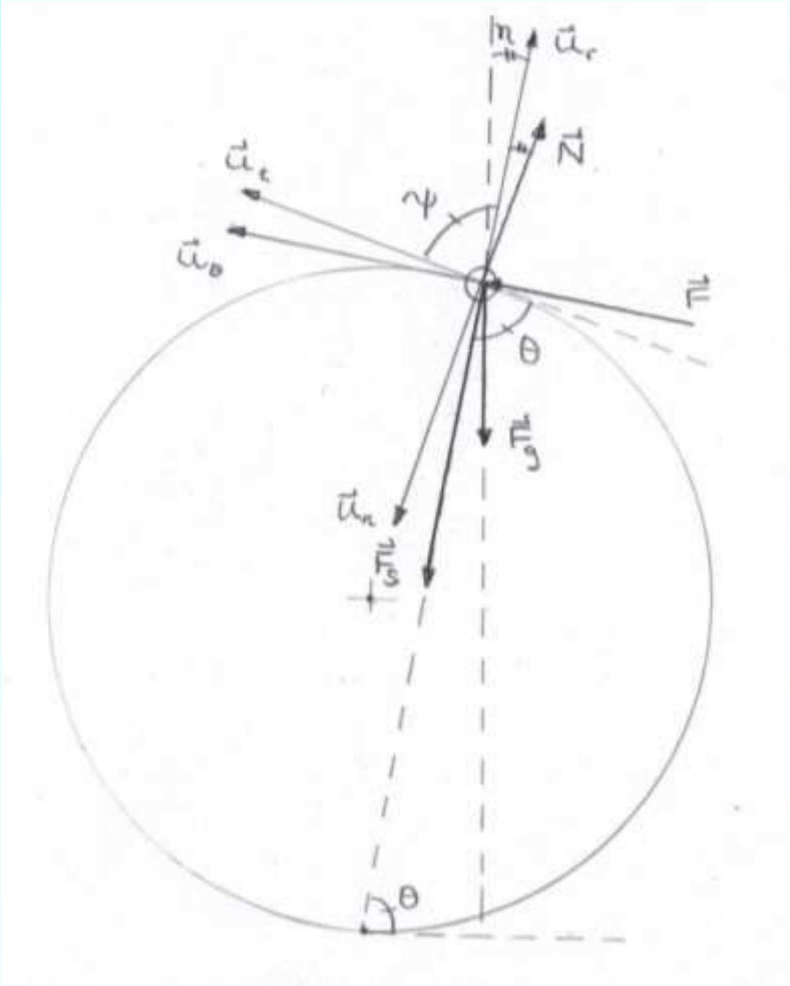
Forces

$$\vec{F} = F\,\vec{u}_\theta$$

$$\vec{N} = -N\,\vec{u}_n = N(\cos\eta\,\vec{u}_r - \sin\eta\,\vec{u}_\theta)$$

$$\vec{F}_s = -k(r - r_0)\vec{u}_r$$

$$\vec{F}_g = -mg\;\vec{j} = -mg(\cos\eta\,\vec{u}_r + \sin\eta\,\vec{u}_\theta)$$



Equations of motion

$$\sum F_r = ma_r : \qquad N\cos\eta - k(r - r_0) - mg\cos\eta = ma_r \qquad (1)$$

$$\sum F_\theta = ma_\theta : \qquad F - N\sin\eta - mg\sin\eta = ma_\theta \qquad (2)$$

Solution of equations of motion

From (1)

$$N = \frac{k(r - r_0) + mg\cos\eta + ma_r}{\cos\eta} = 11.6 \text{ N}$$

From (2)

$$F = N\sin\eta + mg\sin\eta + ma_\theta = 6.24 \text{ N}$$

Chapter 14: Kinetics of a Particle
Work and Energy

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14_COC01

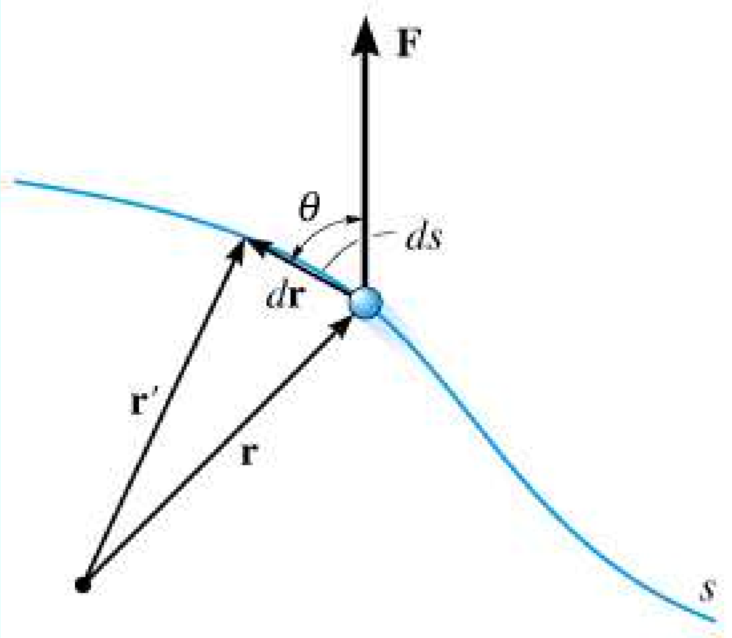
As the woman falls, her energy will have to be absorbed by the bungee cord. The principles of work and energy can be used to predict the motion.


Conserve energy
Turn off the lights
Jokes on you,
Energy is always conserved

14.1 The Work of a Force

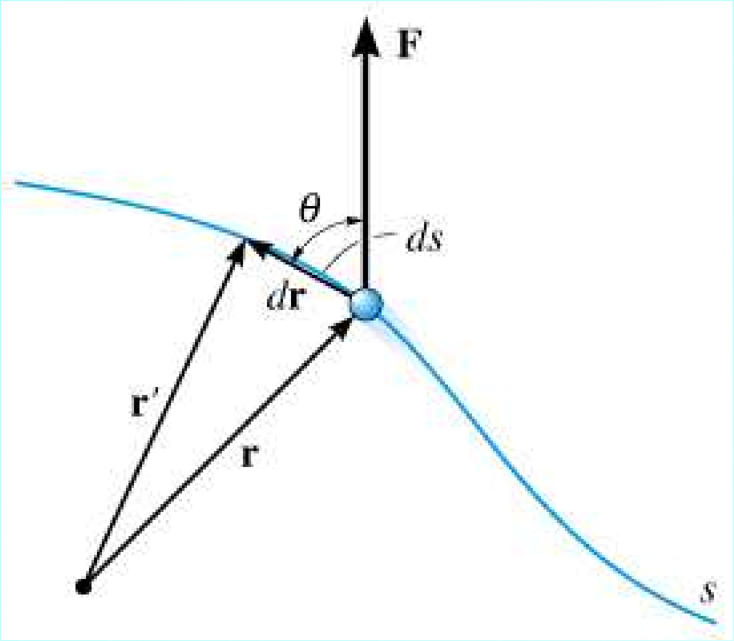
- Now want to develop techniques to analyze motion using the concepts of **work** and **energy**

- We first need to define the concept of the work of a force



- **KEY POINT:** A force, **F**, does **work** on a particle only when the particle undergoes a **displacement in the direction of the force**

- Consider the figure opposite where a particle is travelling along a path, with *s* measuring distance along the path



- As shown in the figure, we consider the situation where the particle's position changes from $\mathbf{r}$ to $\mathbf{r}'$, so that the displacement is

$$d\mathbf{r} = \mathbf{r}' - \mathbf{r}$$

and the magnitude of the displacement is ds

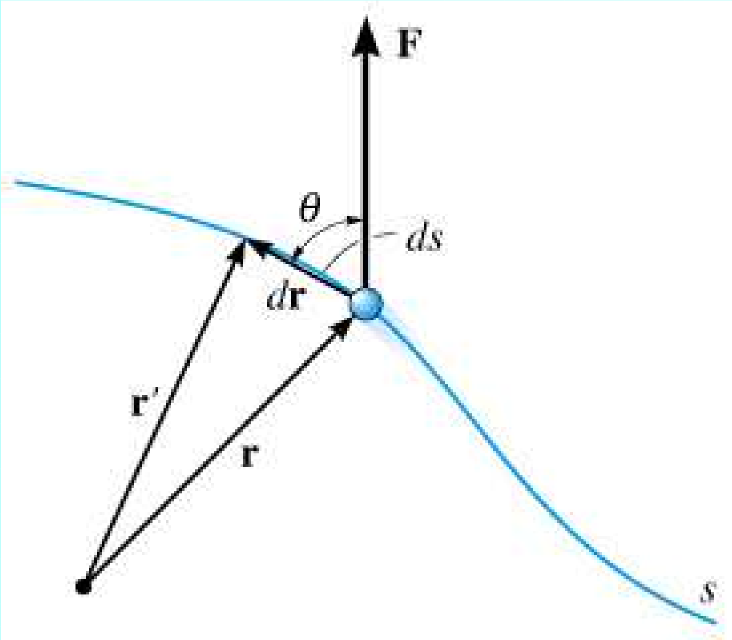
- Then the (infinitesimal) work dU done by $\mathbf{F}$ over this interval is a scalar quantity given by

$$dU = F ds \cos\theta$$

where θ is the angle between $d\mathbf{r}$ and $\mathbf{F}$

- Recalling the definition of the dot product between two vectors ($\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$), we can write the above equation as

$$dU = \mathbf{F} \cdot d\mathbf{r}$$



- Have two ways of interpreting $dU = F ds \cos\theta$

  1. Product of F and the component of the displacement in the direction of $\mathbf{F}$, i.e. $ds \cos\theta$

  2. Product of ds and the component of the force in the direction of $d\mathbf{r}$, i.e. $F \cos\theta$

- **IMPORTANT**: dU can be positive, negative or 0 as follows

  1. $0^\circ \leq \theta < 90^\circ$: $\cos\theta > 0$: Force and displacement vectors have **same** sense, work is **positive**

  2. $90^\circ < \theta \leq 180^\circ$: $\cos\theta < 0$: Force and displacement vectors have **opposite** sense; work is **negative**

  3. $\theta = 90^\circ$: $\cos\theta = 0$: Force and displacement vectors are perpendicular; **no work is done by force**

WORK OF A FORCE (continued)

- Also note that no work is done by the force if there is no displacement

Units for Work

1. **SI: Joule (J)** – 1 Joule is the work done by a force of 1 Newton through a displacement of 1 meter along its line of action, i.e. 1 J = 1 N · m

2. **FPS: ft · lb** – Work done by a force of 1 lb through a displacement of 1 ft along its line of action

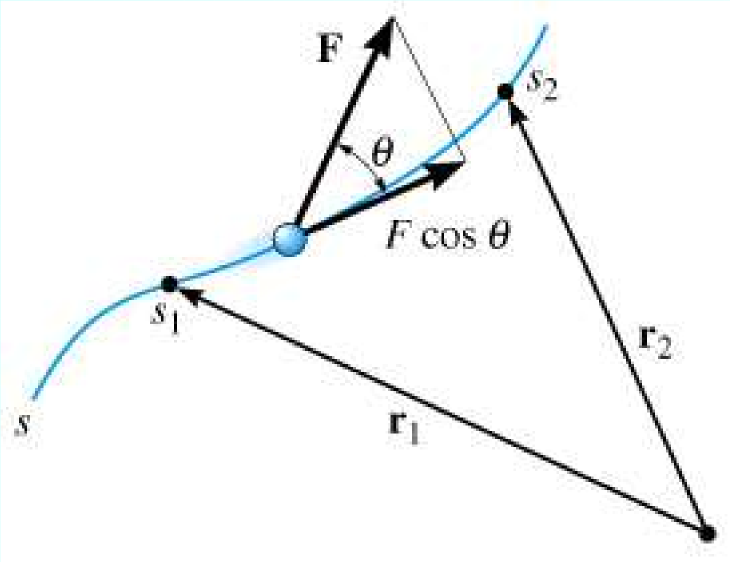
- Note that although the unit combination for work is the same as that for a moment of a force, the concepts of work and moment are **not** to be identified (for example, work is a **scalar** quantity, moment is a **vector**)

WORK OF A FORCE (continued)

- We now consider the work done a force in various circumstances where the displacement is **finite** rather than infinitesimal

- This will generally require **integration** of the basic formula $dU = \mathbf{F} \cdot d\mathbf{r}$

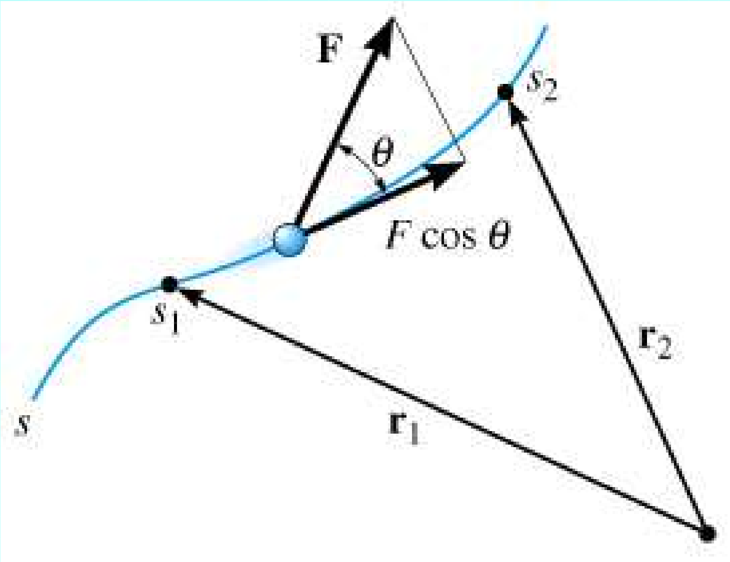
WORK OF A VARIABLE FORCE



- The most general case we can consider is one where the force vector varies (in magnitude, direction, or both), as the particle moves along its path

- Referring to the diagram, the particle moves along the path from s_1 (position vector $\mathbf{r}_1$) to s_2 (position vector $\mathbf{r}_2$)

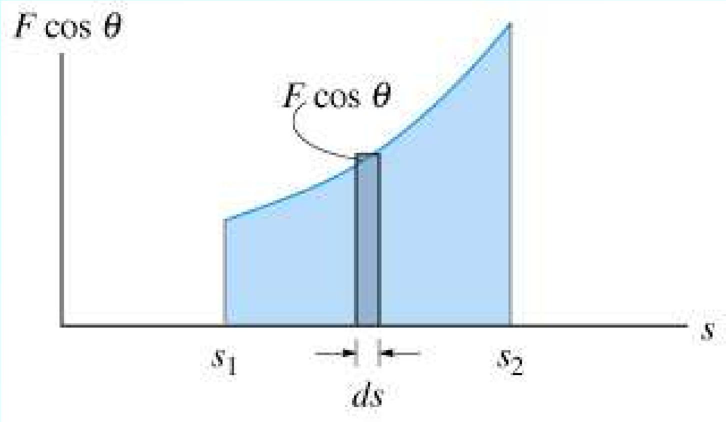
WORK OF A VARIABLE FORCE (continued)



- Assume that the force is a given as a function of s, $\mathbf{F} = \mathbf{F}(s)$

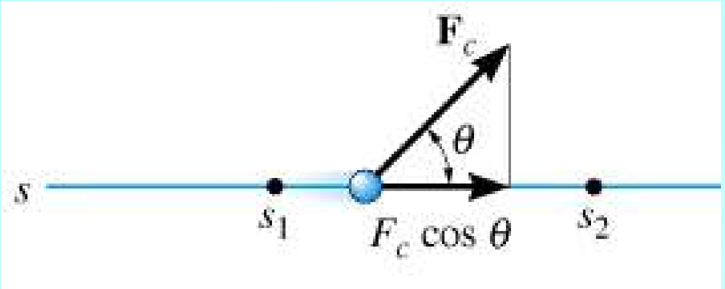
- Then we denote the **finite** amount of work done by the force as U_{1-2} , and compute it using

$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F \cos\theta \, ds$$



- As shown in the figure, and using the usual interpretation of a definite integral, U_{1-2} can be interpreted as the area under the curve of the function $F\cos\theta$ (the working component of the force) from position s_1 to position s_2

WORK OF A CONSTANT FORCE ALONG A STRAIGHT LINE



- Now consider the special, but frequently occurring case where **F** is a constant vector, $\mathbf{F} = \mathbf{F}_c$, acting at a constant angle θ from the path of the particle which is moving in a straight line as shown in the figure at left

- Then the work done by the force as the particle moves from s_1 to s_2 is

$$U_{1-2} = F_c \cos\theta \int_{s_1}^{s_2} ds$$

.

or

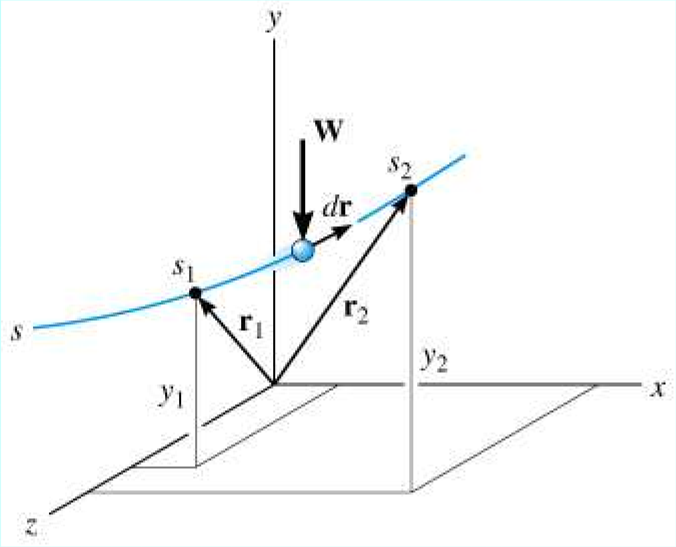
$$U_{1-2} = F_c \cos\theta (s_2 - s_1)$$

which we can also write as

$$U_{1-2} = F_c \cos\theta \, \Delta s$$

WORK OF A CONSTANT FORCE ALONG A STRAIGHT LINE (continued)

- In this case the area under the $F\cos\theta$ curve is simply the area of a rectangle, as shown in the figure



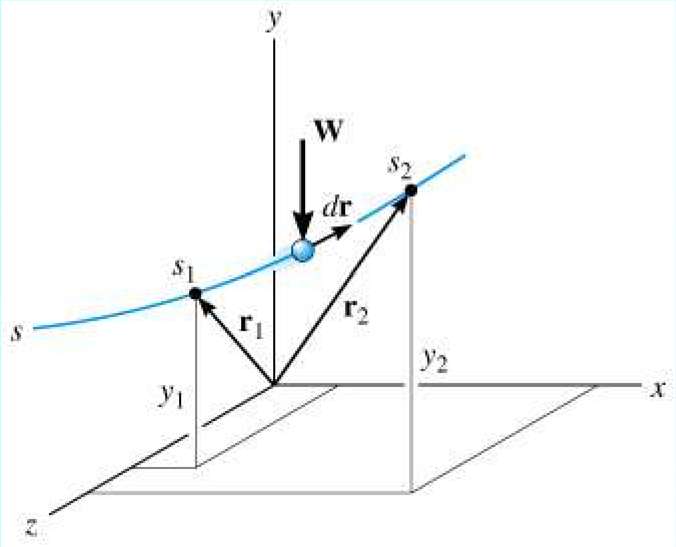
WORK OF A WEIGHT

- Consider a particle with weight, **W**, moving **up** along a path parameterized by s from position s_1 to s_2 as shown in the figure (note the orientation of the axes, such that y is the vertical direction)

- At any point along the path, the infinitesimal displacement, $d\mathbf{r}$, is given by

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

WORK OF A WEIGHT (continued)



- We have $\mathbf{W} = -W\,\mathbf{j}$, so the work done by the weight is

$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} (-W\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= -\int_{y_1}^{y_2} W\,dy = -W(y_2 - y_1)$$

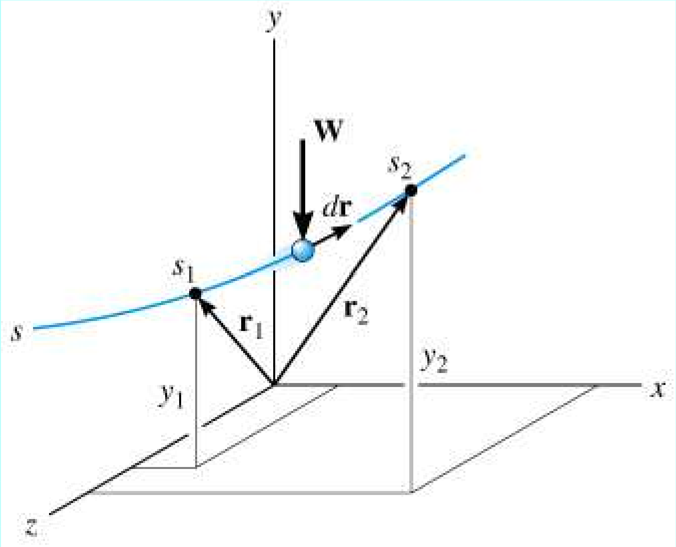
.

or

$$U_{1-2} = -W\Delta y$$

- Note that the work done by the weight is **independent** of the path taken between the initial and final positions of the particle

WORK OF A WEIGHT (continued)

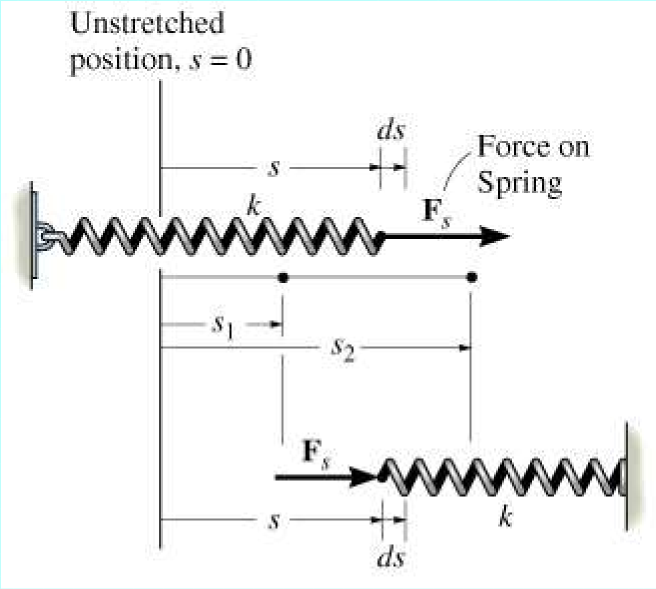


- **IMPORTANT!** Always need to be careful with signs when dealing with work. In this case we have

$$U_{1-2} = -W\Delta y$$

1. Particle moves **upward** ($\Delta y > 0$), weight and displacement are in **opposite** directions, work is **negative**

2. Particle moves **downward** ($\Delta y < 0$), weight and displacement are in **same** direction, work is **positive**



WORK OF A SPRING FORCE

- Here we first consider the work done **on** a spring by a force that elongates/compresses it, then the work done on a particle attached to the spring, i.e. by the force exerted **by** the spring

- **Recall**: Magnitude of force exerted on spring displaced distance *s* from its equilibrium position is

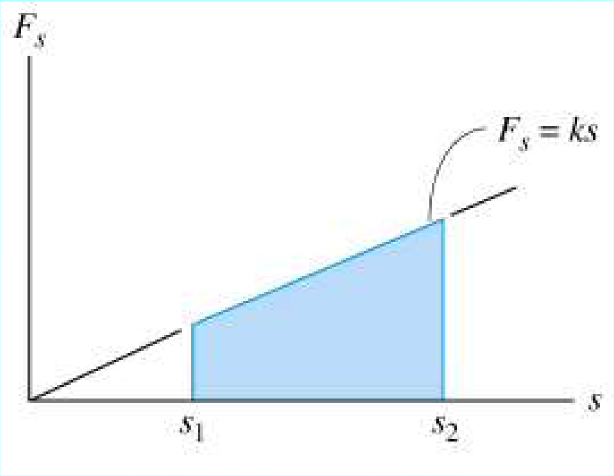
$$F_s = ks$$

where *k* is the spring constant (stiffness)

.

- Referring to the figure above, if the spring is initially elongated (or compressed) a distance s_1 from equilibrium, and then is **further** elongated (or compressed) to a distance s_2 from equilibrium, then the work done is **positive**, since the force on the spring and the displacement are in the same direction

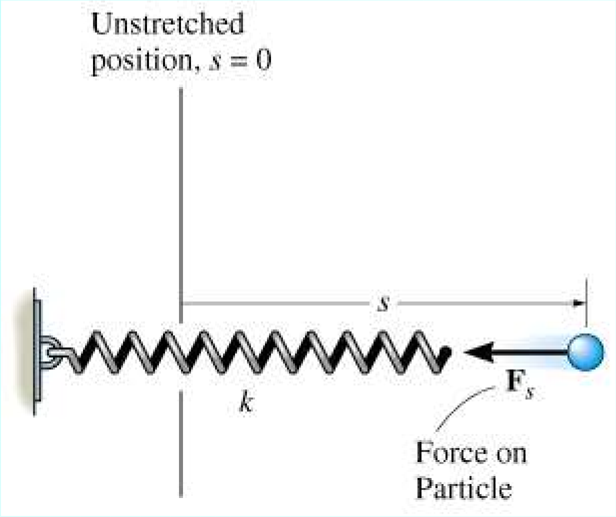
WORK OF A SPRING FORCE (continued)



- Thus, we have

$$U_{1-2} = \int_{s_1}^{s_2} F_s\, ds = \int_{s_1}^{s_2} ks\, ds = \frac{1}{2} k s_2^{\,2} - \frac{1}{2} k s_1^{\,2}$$

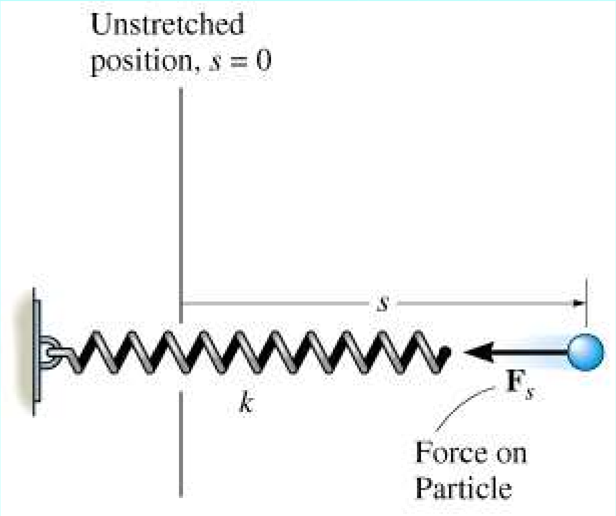
which can be interpreted as the trapezoidal area under the graph $F_s = ks$ as shown in the figure



- Now consider the second case (and the most relevant one for problem solving) where a particle is attached to the spring

- In accordance with Newton's third law, the force, $\mathbf{F}_s$ on the particle now always acts in the **opposite** direction of the displacement (again assuming that we are either further stretching or further compressing the spring)

WORK OF A SPRING FORCE (continued)



- Thus, the work done by the force on the particle is **negative** and given by

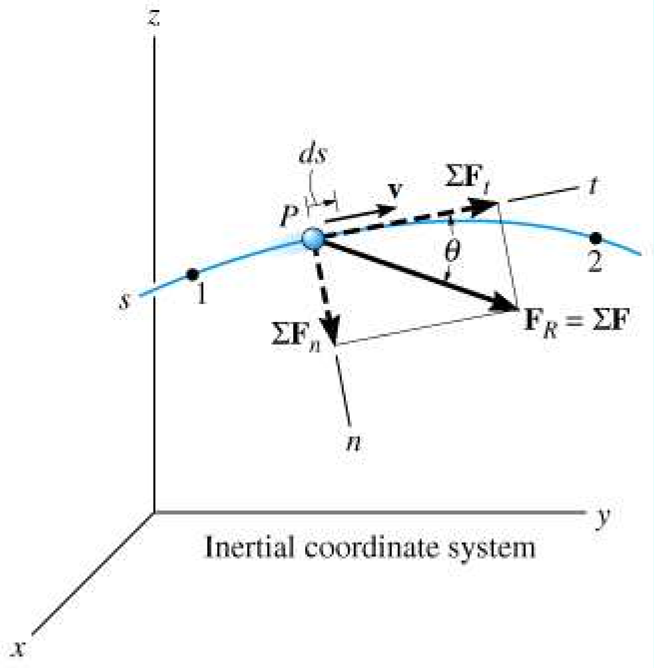
$$U_{1-2} = -\left(\frac{1}{2}ks_2^{\,2} - \frac{1}{2}ks_1^{\,2}\right)$$

IMPORTANT: We again need to be careful with signs when computing work

1. If the force of the spring on the particle and the particle displacement are in the same direction, the work done by the spring force is **positive**

2. If the force of the spring on the particle and the particle displacement are in opposite directions, the work done by the spring force is **negative**

14.2 Principle of Work and Energy

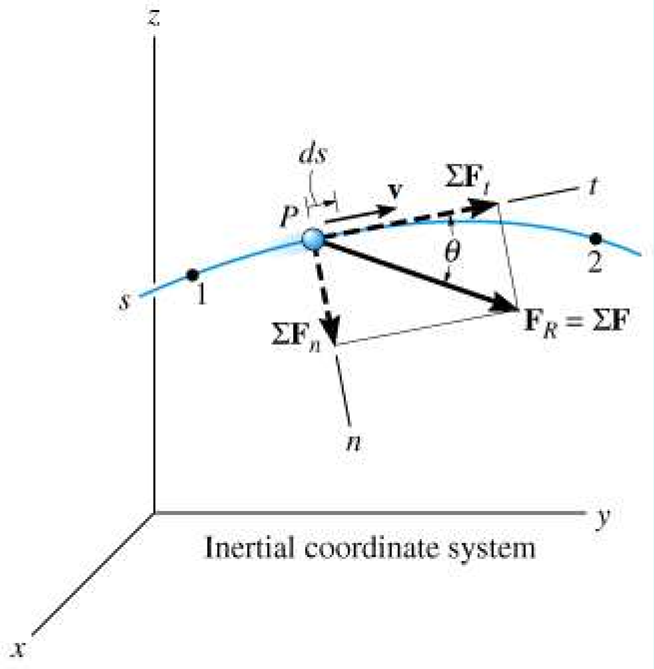


- Consider the figure at the left which shows a particle moving along some path with position, velocity, acceleration and forces acting on it defined with respect to an inertial coordinate system

- At some instant of time, the particle is at location P, with resultant force $\mathbf{F}_R = \Sigma \mathbf{F}$ acting on it

- We can introduce tangential and normal coordinates (t,n) at P and consider the EOM in the t direction

$$\Sigma F_t = m a_t$$



PRINCIPLE OF WORK & ENERGY (cont.)

- Now, recall that for tangential motion we can apply the kinematic equation

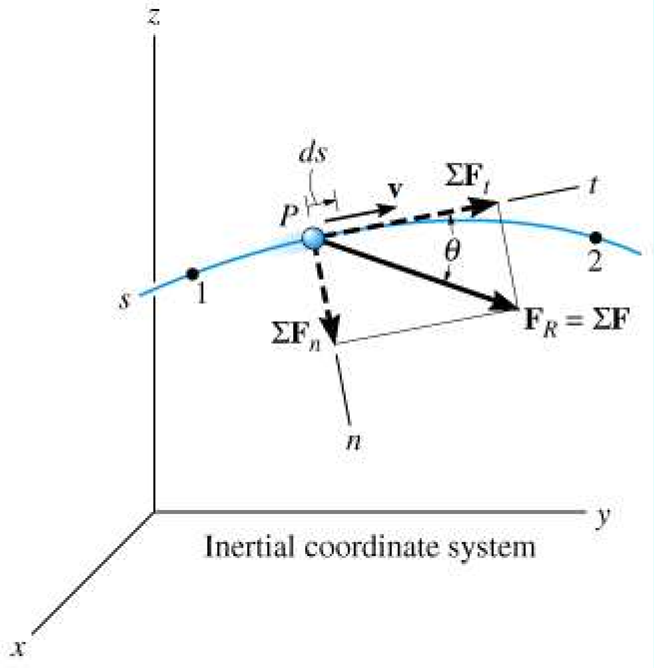
$$a_t\, ds = v\, dv$$

- We multiply both sides of this equation by m and integrate along the path from

1. Where $s = s_1$ and $v = v_1$, to
2. Where $s = s_2$ and $v = v_2$

- We thus have

$$\int_{s_1}^{s_2} m a_t\, ds = \int_{v_1}^{v_2} m v\, dv$$



PRINCIPLE OF WORK & ENERGY (cont.)

- We can manipulate this last equation as follows

$$\int_{s_1}^{s_2} m a_t \, ds = \int_{v_1}^{v_2} m v \, dv$$

$$\int_{s_1}^{s_2} (\Sigma F_t)\, ds = \frac{1}{2} m v_2^{\,2} - \frac{1}{2} m v_1^{\,2}$$

- Now we note from the figure that $\Sigma F_t = \Sigma F \cos\theta$, so that the term on the left is simply the **total work done by all forces acting on the particle from point 1 to point 2 on the path**

- We have thus derived one form of the principle of work and energy (PWE)

$$\Sigma U_{1-2} = \frac{1}{2} m v_2^{\,2} - \frac{1}{2} m v_1^{\,2}$$

PRINCIPLE OF WORK & ENERGY (continued)

- We recognize the terms of the form $mv^2/2$ as representing the **kinetic energy** of the particle, which we will denote by T

$$T = \frac{1}{2}mv^2 = \text{Kinetic energy of particle}$$

- **NOTE:** Units of T = units of U = Joule (J)

- We can thus recast the principle of work and energy in the form

$$T_1 + \Sigma U_{1-2} = T_2$$

- In words: The initial kinetic energy of the particle plus the work done by all of the forces acting on the particle as it moves from its initial to final position is equal to the final kinetic energy of the particle

USE of PRINCIPLE OF WORK & ENERGY IN PROBLEM SOLVING

- The principle is an integrated version of $\Sigma F_t = ma_t$
- It can thus often be used instead of $\Sigma F_t = ma_t$ in problems that involve forces, velocities and displacements, and may save calculational steps, particularly if final (initial) speeds of particles are to be determined
- Note that the principle cannot replace **all** equations of motion – for example, it cannot be used to compute **general** forces that are normal to the particle path since those forces **do not** do work on the particle
- However, for motion on known curved paths, where we have $\Sigma F_n = mv^2 / \rho$, we may be able to use the principle to compute *v*, then compute the normal force, and this may easier than using the "standard" formulae