PHYS 170 Section 101
Lecture 19
October 22, 2018

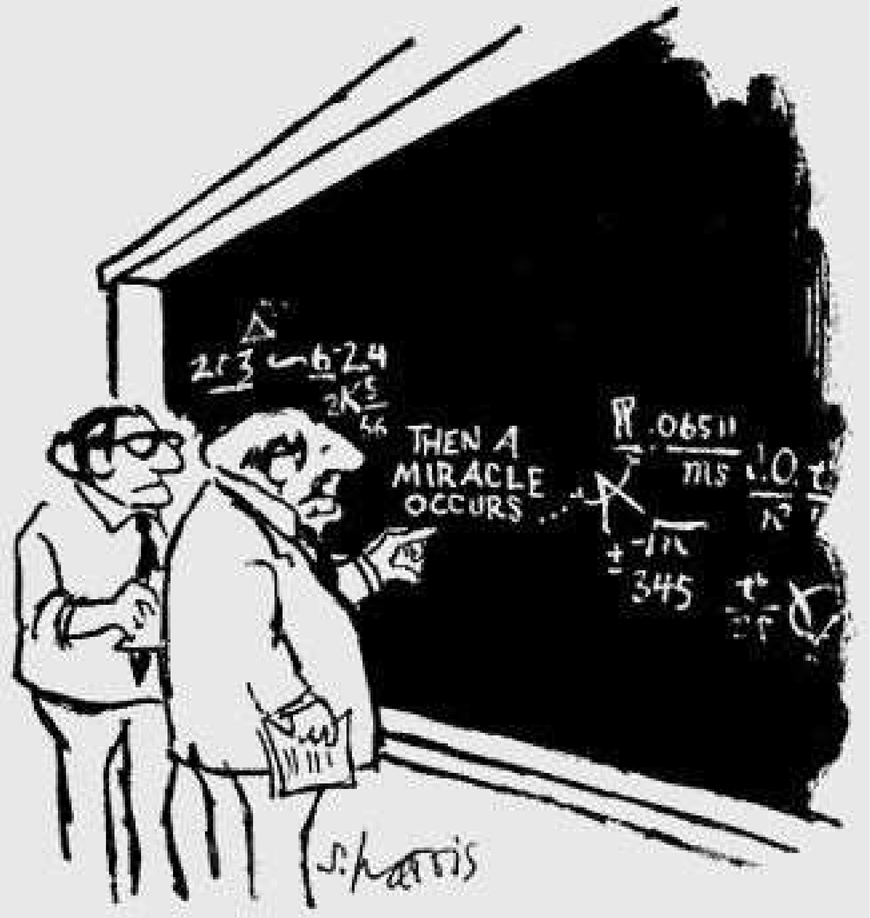
Oct 22—Announcements

- Homework assignment 6 is due tonight at 11:59 PM
- Homework assignment 7 is due this Friday at 11:59 PM

Lecture Outline/Learning Goals

- Two worked problems of curvilinear motion using tangential and normal components
- 12.8 Curvilinear Motion: Cylindrical (Polar) Components

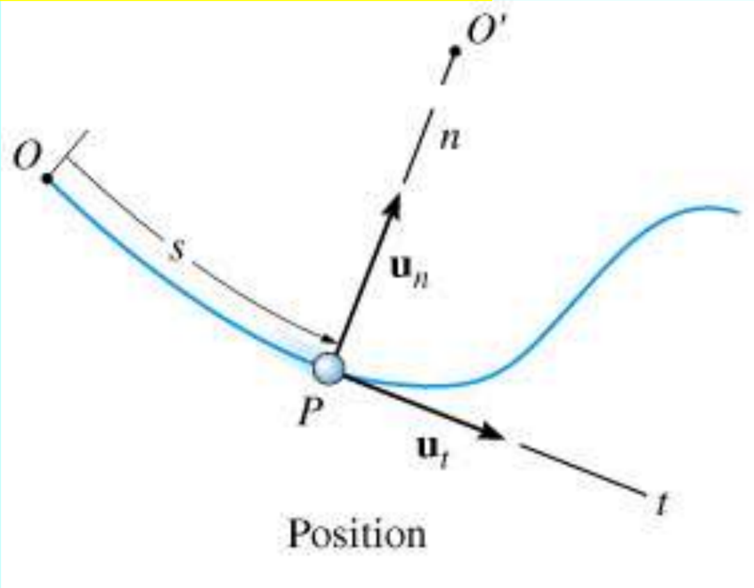
Perhaps the last lecture left you feeling a bit like this …



"I think you should be more explicit here in step two."

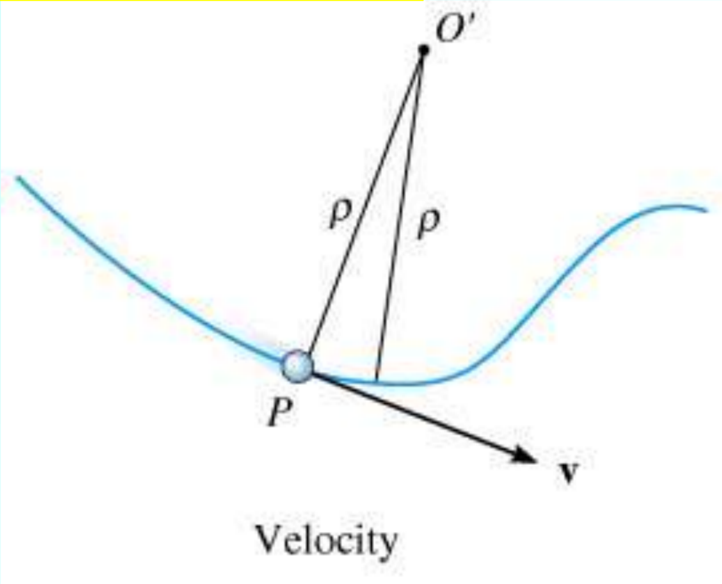
Curvilinear Motion: Normal & Tangential Components

Center of curvature, O'



(a)

Radius of curvature, ρ



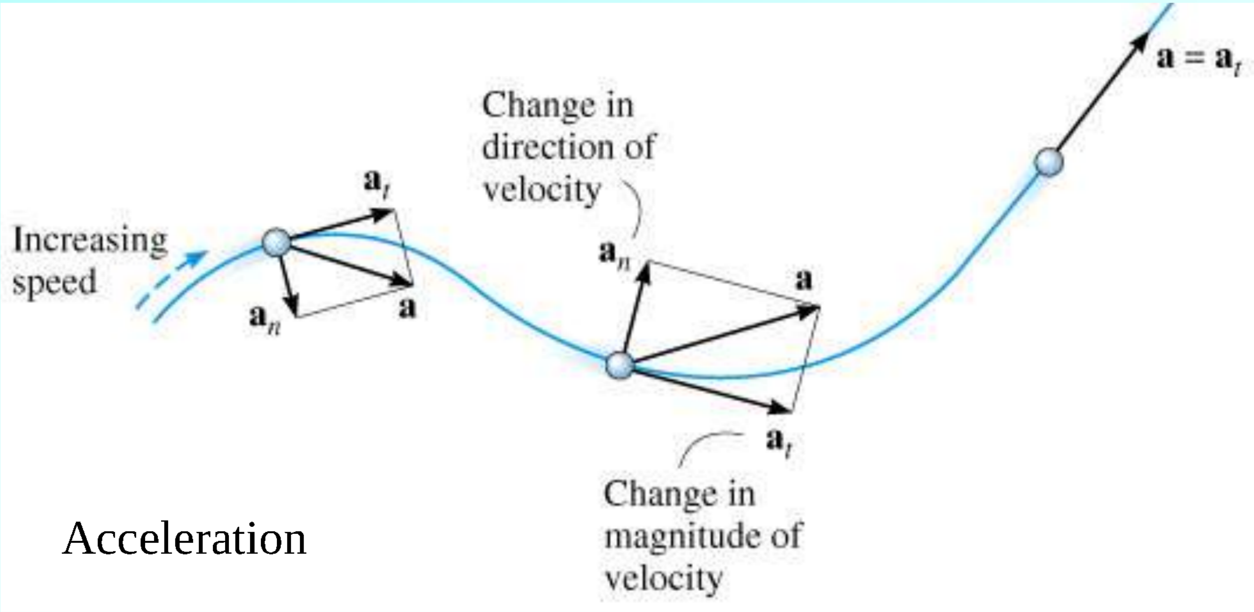
(c)

Position: $s = s(t)$ as measured from O

$$\mathbf{v} = v\,\mathbf{u}_t$$

$$v = \frac{ds}{dt} = \dot{s}$$

Curvilinear Motion: Normal & Tangential Components



$$\mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n$$

$$a_t = \dot{v} \qquad \text{or} \qquad a_t\, ds = v\, dv$$

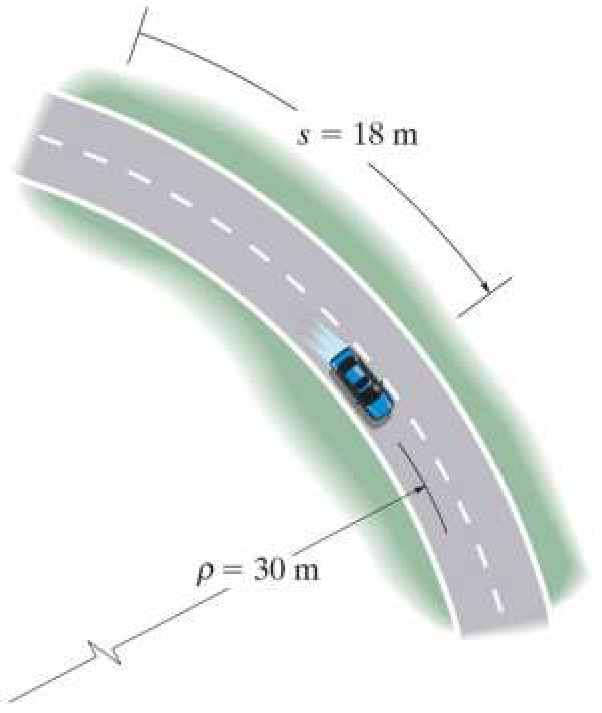
$$a_n = \frac{v^2}{\rho}$$

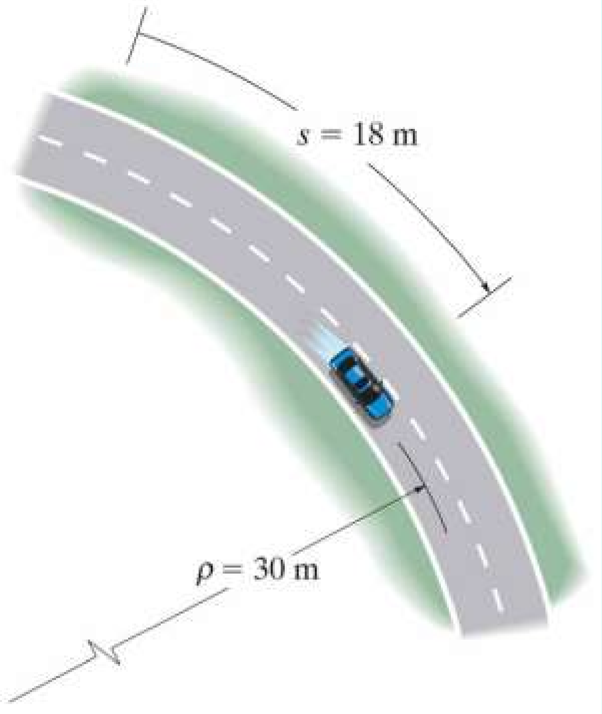
Problem 12-120 (page 66, 14th edition)

The car travels along the circular path. Starting from rest, its acceleration along the path is $0.5e^t$ m/s^2 where t is in seconds.

(1) Determine how long it takes the car to travel 18 m.

(2) Determine the car's speed and acceleration at this time.





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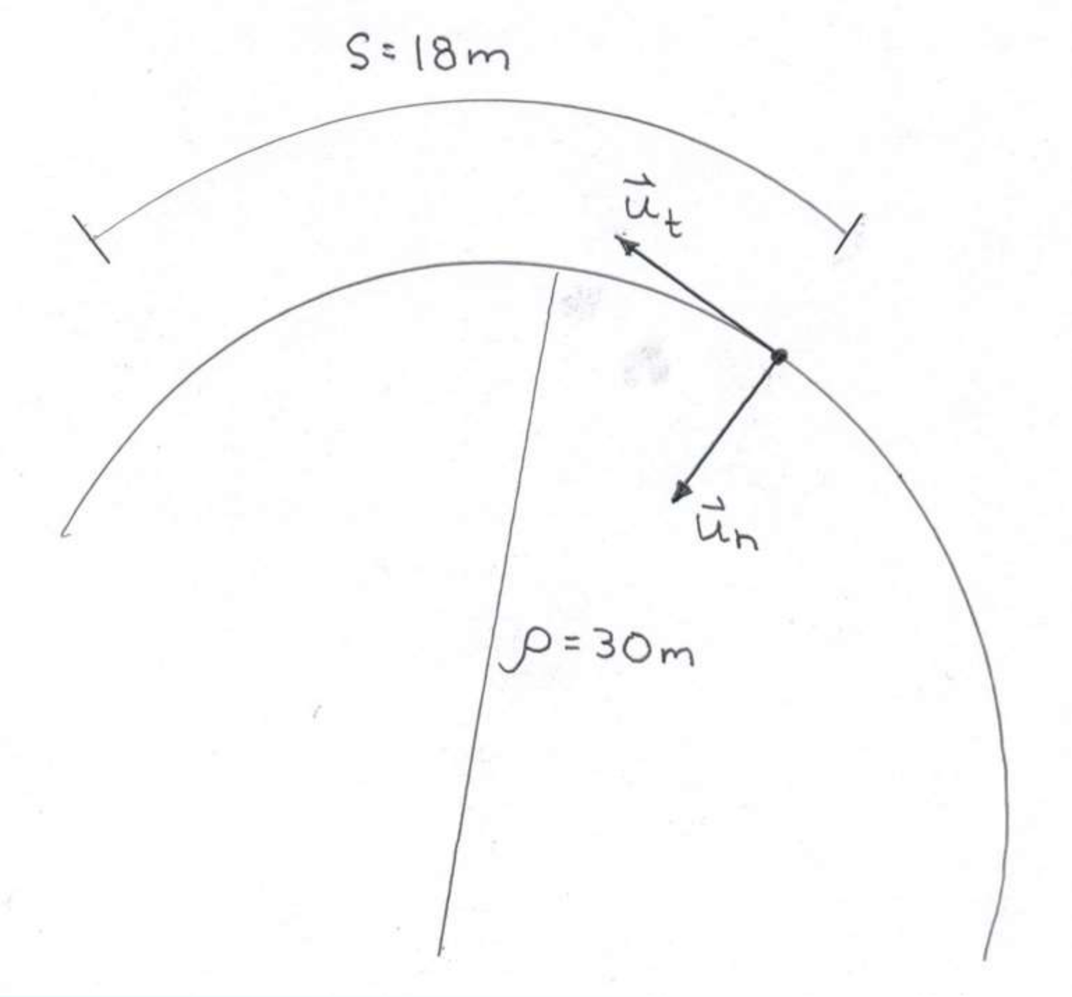
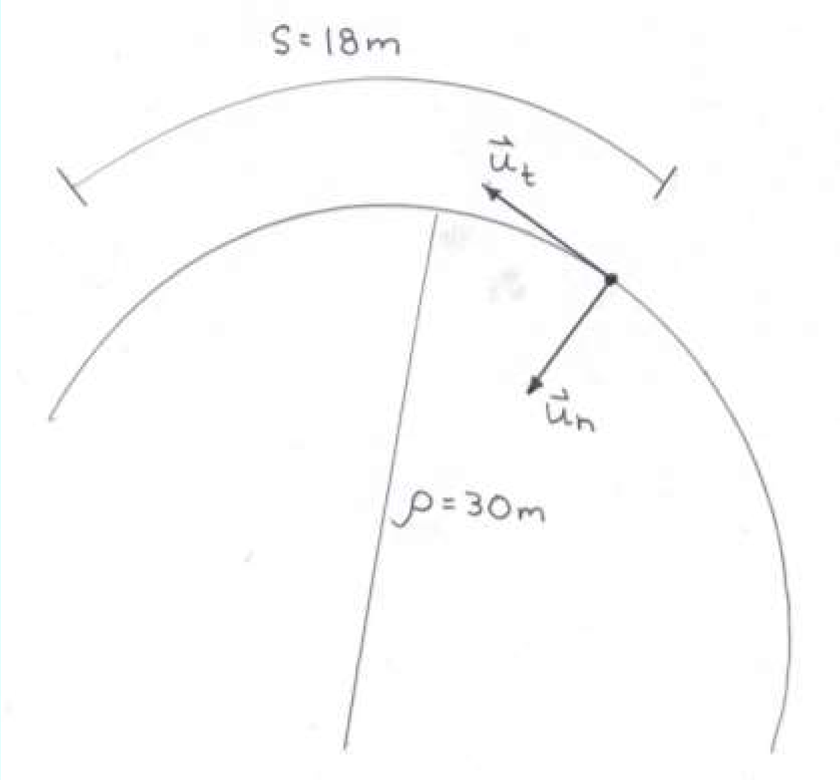




Solution strategy

(1) Integrate tangential acceleration (acceleration along the path) to determine the car's speed, $v(t)$

(2) Integrate the speed $v(t)$ to determine the distance traveled, $s(t)$, in time t.

(3) Determine the time t at which the car has traveled 18 m.

(4) Determine the car's speed and acceleration at that time.

The car's speed $v(t)$ is determined by integrating its tangential acceleration

$$a_t = \frac{dv}{dt}$$

$$\int_0^v dv = \int_0^t a_t dt = \int_0^t 0.5e^t dt$$

$$v(t) = 0.5e^t \Big|_0^t = 0.5\left(e^t - 1\right) \text{ m/s}$$

The distance $s(t)$ the car travels in time t is determined by integrating its speed $v(t)$

$$v = \frac{ds}{dt}$$

$$\int_0^s ds = \int_0^t v(t)dt = \int_0^t 0.5\left(e^t - 1\right)dt$$

$$s(t) = 0.5(e^t - t)\Big|_0^t = 0.5(e^t - t - 1) \text{ m}$$

The time t when the car has travelled 18 m along the path is determined by solving

$0.5(e^t - t - 1) = 18$ or $0.5(e^t - t - 1) - 18 = 0$

Using, for example, the solver function on a TI graphing calculator, we find

$t = 3.7064 \text{ s}$

so it takes 3.71 s for the car to travel 18 m.

At this time we have the following:

$$v = 0.5(e^t - 1) = 19.9 \text{ m/s}$$

$$a_t = 0.5\, e^t = 20.4 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = 13.1 \text{ m/s}^2$$

$$a = \sqrt{a_t^{\ 2} + a_n^{\ 2}} = 24.2 \text{ m/s}^2$$

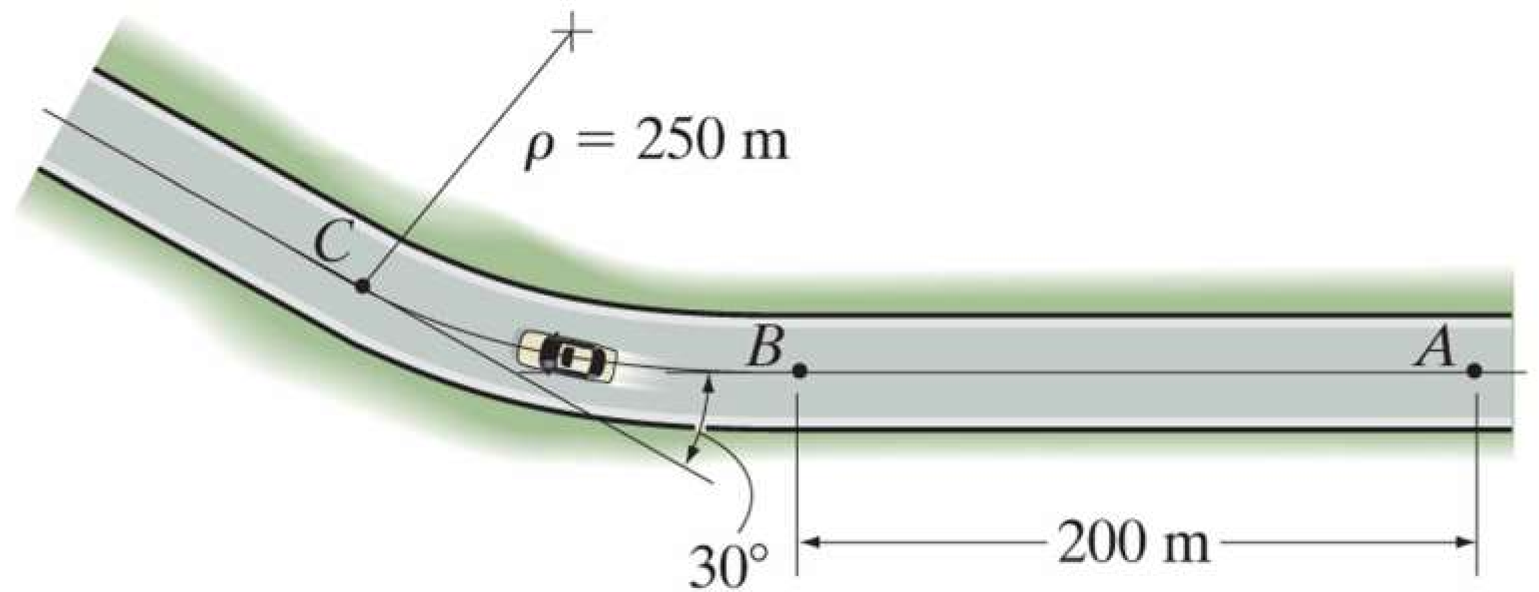
Note: These values, and those in subsequent problems of this type were computed using full calculator precision for dependent values (t, v, etc.) and then rounded

Problem 12-125 (page 62, 12th edition)

The car is travelling at 25 m/s at *A*. The brakes are applied at *A* and its speed is reduced by $t^{1/2}$ / 4 m/s^2 where *t* is in seconds.

(1) Determine how long it takes the car to travel from *A* to *C*.

(2) Determine the car's speed and acceleration when it reaches *C*.



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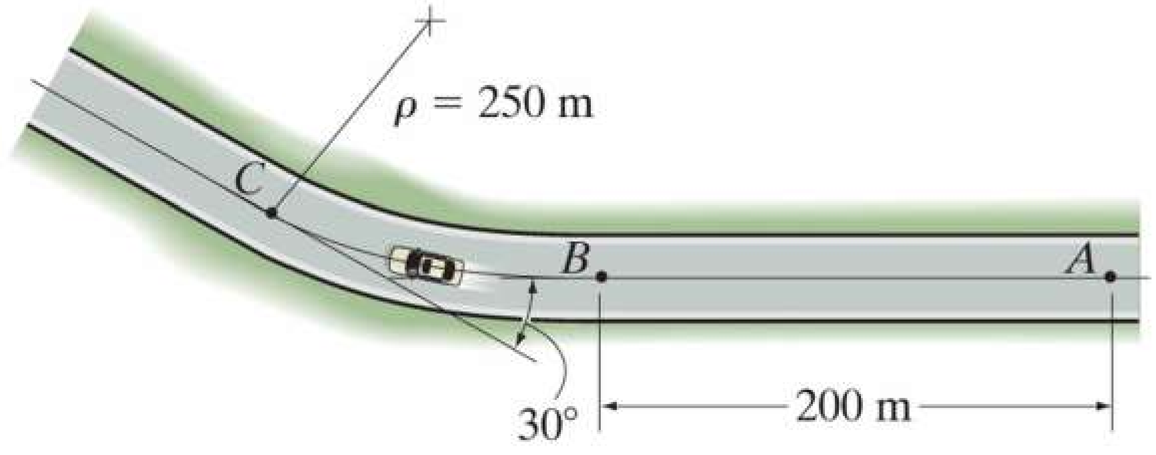

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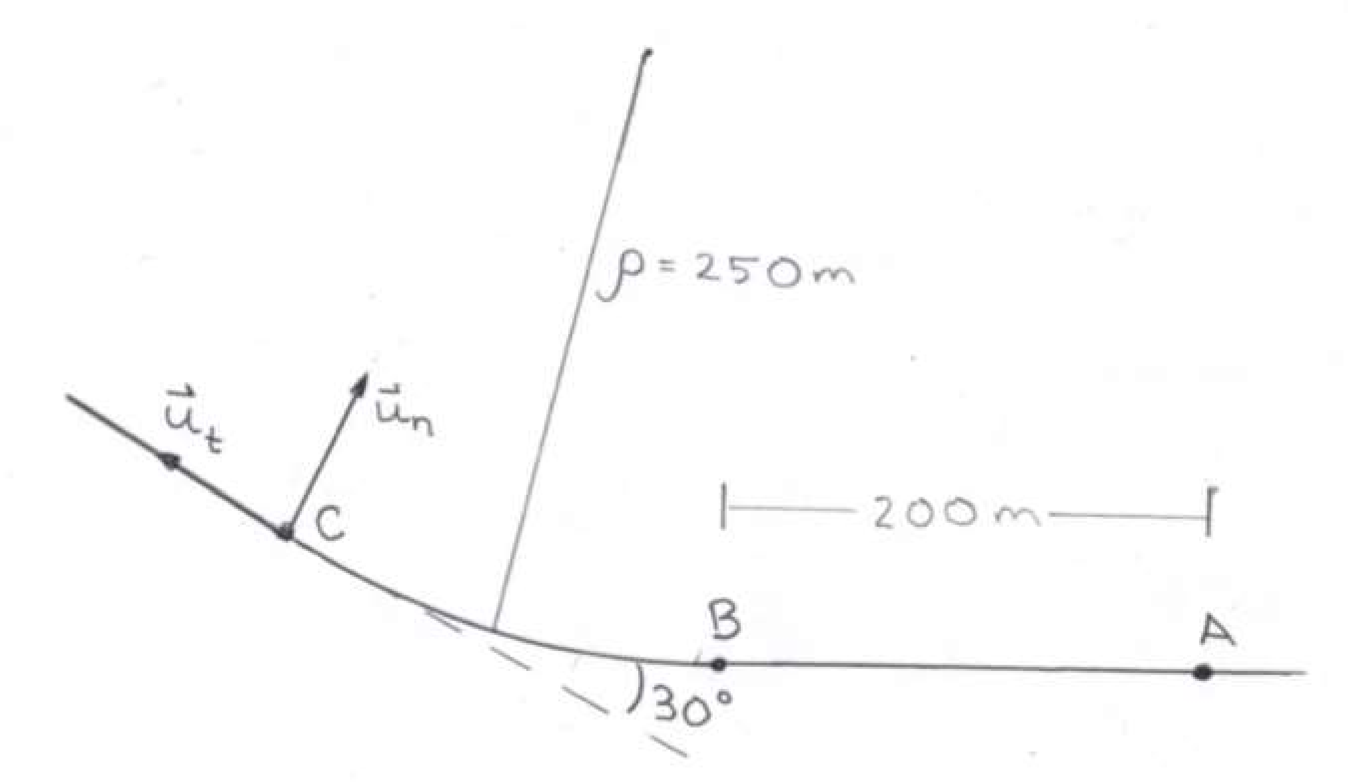
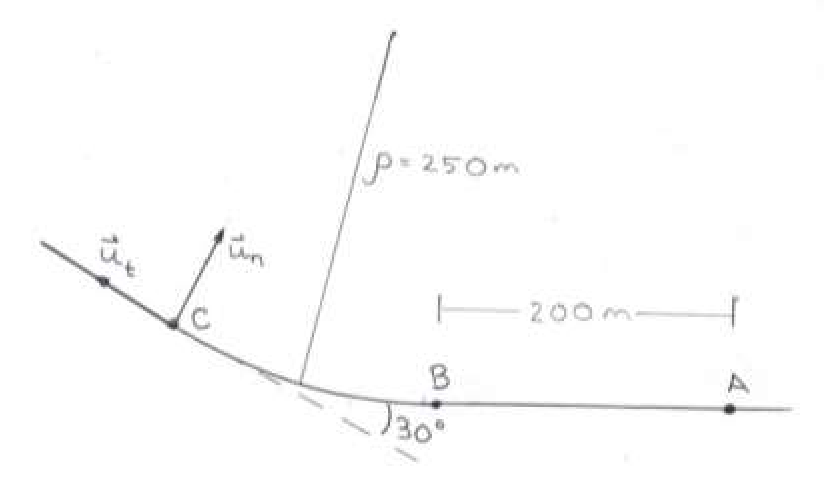




Solution strategy

(1) Integrate tangential acceleration (acceleration along the path) to determine the car's speed, $v(t)$

(2) Integrate the speed $v(t)$ to determine the distance traveled, $s(t)$, in time t.

(3) Determine the distance from A to C using given geometrical information.

(4) Determine time t needed to travel that distance.

(5) Determine the car's speed and acceleration at that time.

Starting from A, the car's speed $v(t)$ at time t along the path is determined by integrating the car's tangential acceleration

$$a_t = \frac{dv}{dt}$$

$$\int_{v_0}^{v} dv = \int_0^t a_t dt$$

$$\int_{25}^{v} dv = \int_0^t \left(-t^{1/2}/4\right) dt$$

$$v - 25 = -\frac{1}{4}\frac{t^{3/2}}{3/2}\Bigg|_0^t = -\frac{t^{3/2}}{6}$$

so

$$v(t) = 25 - \frac{t^{3/2}}{6} \text{ m/s}$$

The distance $s(t)$ the car travels in time t from A is determined by integrating its speed $v(t)$

$$v = \frac{ds}{dt}$$

$$\int_0^s ds = \int_0^t v(t)dt = \int_0^t \left(25 - t^{3/2}/6\right)dt$$

$$s = \left(25t - \frac{1}{6}\frac{t^{5/2}}{5/2} \right)\Bigg|_0^t$$

so

$$s(t) = 25t - \frac{t^{5/2}}{15} \text{ m}$$

To determine the distance between A and C, note that the angle subtended by the arc BC is $\Delta\theta = 30^\circ = \pi/6$ radians. Thus, the distance from B to C is $\rho\Delta\theta = 250\pi/6$ m, and the distance from A to C is $200 + 250\pi/6$ m. The time t when the car is at C is determined by solving

$$25t - \frac{t^{5/2}}{15} = 200 + 250\pi/6$$

or

$$25t - \frac{t^{5/2}}{15} - 200 - 250\pi/6 = 0$$

Using solver we find the smallest value of t that satisfies the above equation

$t = 15.94$ s

So it takes 15.9 s for the car to travel from A to C.

At time $t = 15.94$ s we have the following

$$v = 25 - t^{3/2}/6 = 14.4 \text{ m/s}$$

$$a_t = -t^{1/2}/4 = -0.998 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = 0.828 \text{ m/s}^2$$

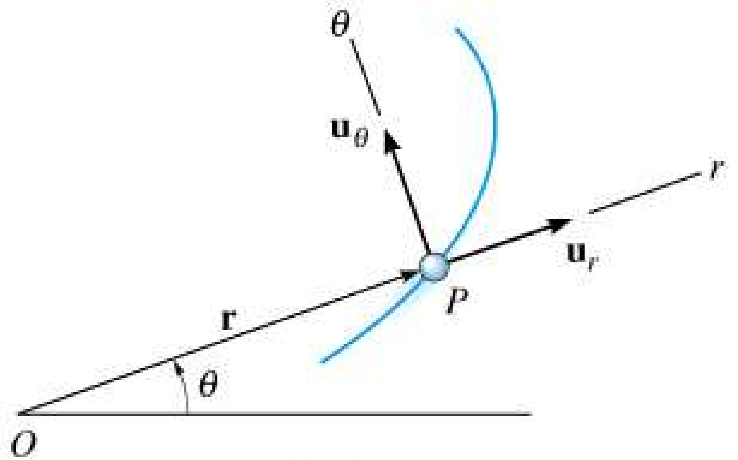
$$a = \sqrt{a_t^{\,2} + a_n^{\,2}} = 1.30 \text{ m/s}^2$$

12.8 Curvilinear Motion: Cylindrical (Polar) Coordinates

MOTIVATION

- For certain types of particle motion, it is often convenient and/or natural to specify position in terms of a **radial distance** and an **angular position**
  - **2D (planar motion): Polar coordinates:** (r,θ)
  - **3D: Cylindrical coordinates:** (r,θ,z)
- As previously, will focus attention on the 2D (planar) case – refer to text for discussion of extension to cylindrical coordinates
- Due to time restrictions and similarity of proofs to those in discussion of tangential / normal components, we will skip some steps in the derivations that follow – see text for additional details

POLAR COORDINATES



Position

(a)

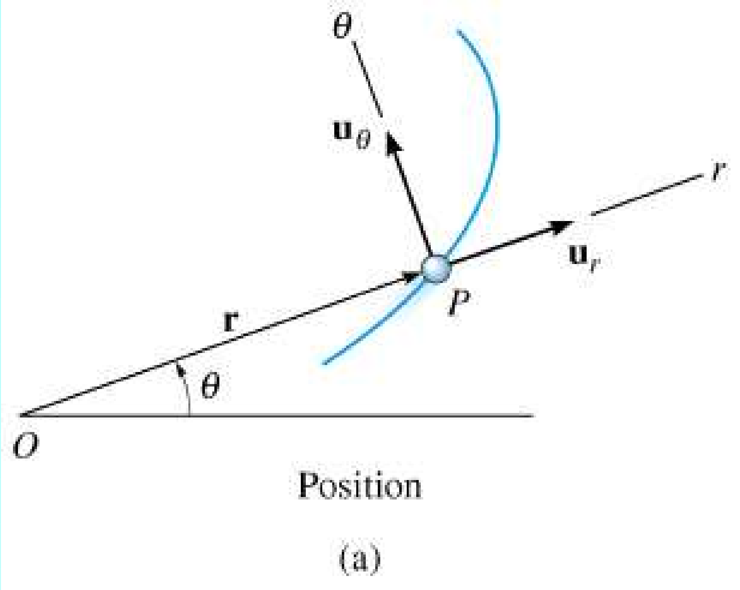
- Again, consider particle located at position, *P*, and traveling along some path, as shown in Fig (a).

- We locate the particle by specifying

  1. Radial coordinate, *r*, measured outwards from the origin, *O*, of the polar coordinate system

  2. Transverse or angular coordinate, θ , measured counterclockwise from a fixed axis through the origin (typically a horizontal axis as in the figure) to the *r* axis

- **NOTE:** Although the angular coordinate can be measured in degrees, the **natural** units are **radians** (recall: 1 radian = $180^\circ / \pi$) and the formulae involving angular velocities and angular accelerations derived below are simplest when the associated units of rad/s and rad/s^2 are used. Thus rad/s and rad/s^2 will generally be used for angular velocity and angular acceleration, although angles themselves will usually still be specified in degrees (so that you **won't** have to switch units on your calculator from degrees to radians)



POLAR COORDINATES

- As usual, with each of the r, θ axes we can associate unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ which are perpendicular to one another

(Positive) direction of $\mathbf{u}_r$: Hold θ fixed;

$\mathbf{u}_r$ points in direction of increasing r

(Positive) direction of $\mathbf{u}_\theta$: Hold r fixed;

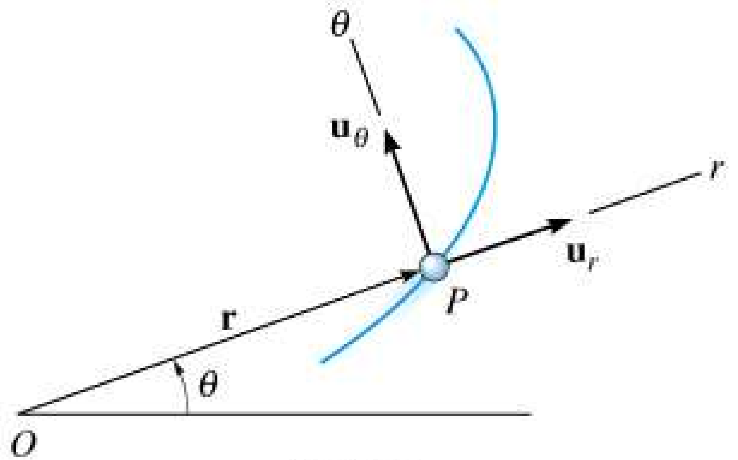
$\mathbf{u}_\theta$ points in direction of increasing θ

- As is the case for (t,n) coordinates, as particle moves along path, the orientations of the unit vectors will change in general. However, the origin, O, remains fixed

- As particle moves, its (r,θ) coordinates will generally be functions of time

$$(r,\theta) = (r(t), \theta(t))$$

- As usual, proceed to derive expressions for the particle's **position**, **velocity** and **acceleration** vectors in these coordinates



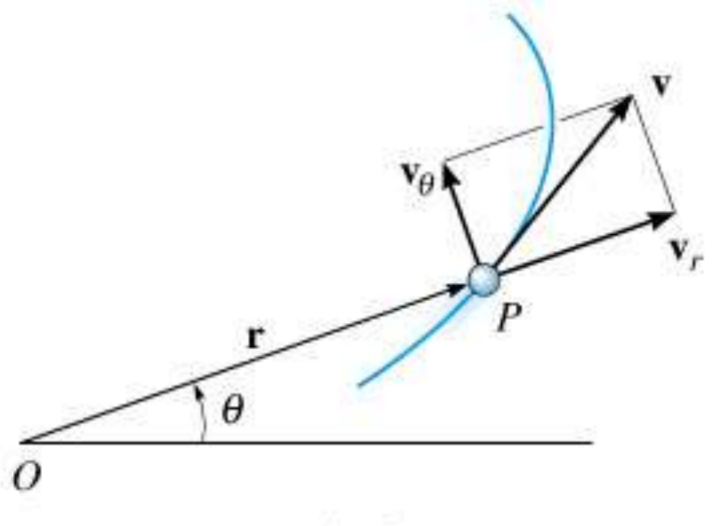
Position

(a)

POSITION

- At any given moment of time, t, when the particle has polar coordinates (r,θ) , the position vector is simply

$$\mathbf{r} = r\,\mathbf{u}_r$$



Velocity

(c)

VELOCITY

- As usual, the instantaneous velocity, **v**, is given by taking the first time derivative of **r**

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{u}_r + r\,\dot{\mathbf{u}}_r$$

- We state without proof (see text for argument), that

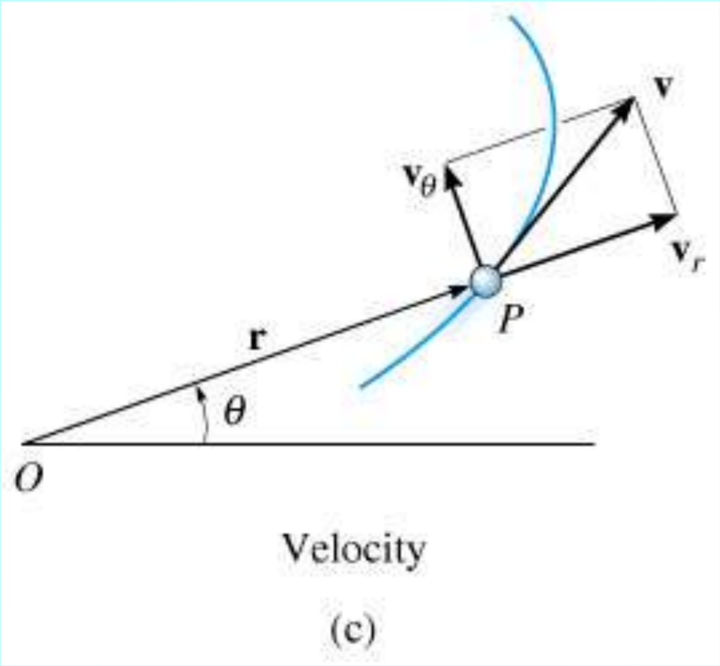
$$\dot{\mathbf{u}}_r = \dot{\theta}\,\mathbf{u}_\theta$$

- We thus have the following formula for the component form of the velocity in polar coordinates

$$\mathbf{v} = v_r\,\mathbf{u}_r + v_\theta\,\mathbf{u}_\theta$$

where the components are given by

$$v_r = \dot{r}$$
$$v_\theta = r\,\dot{\theta}$$



VELOCITY (continued)

- The velocity components have the following names and interpretations

$\mathbf{v}_r = \dot{r}\,\mathbf{u}_r$: Radial component

Time rate of change of radial position

$\mathbf{v}_\theta = (r\dot{\theta})\mathbf{u}_\theta$: Transverse component

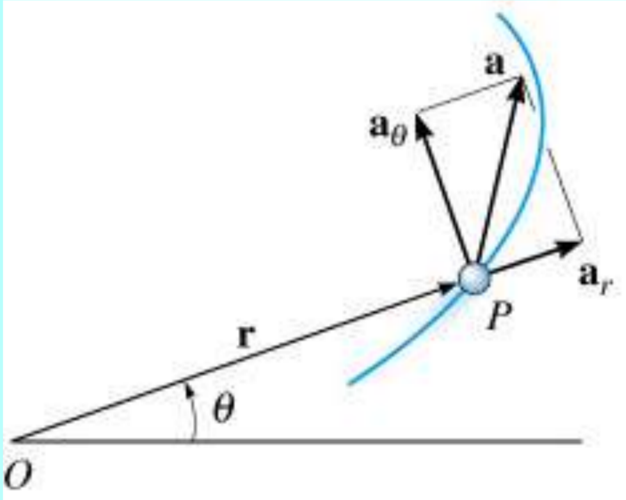
Rate of motion along circumference of circle with radius *r*

- **Nomenclature:** $\dot{\theta} = d\theta / dt$ is known as the **angular velocity** since it quantifies the time rate of change of the angular coordinate, and is measured in **rad/s**

- **Magnitude of velocity:**

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

- **Direction of velocity:** In direction tangent to the particle path at *P*, as usual



Acceleration

(e)

ACCELERATION

- Take the first time derivative of the velocity vector to get the instantaneous acceleration of the particle

$$\mathbf{v} = \dot{r}\,\mathbf{u}_r + (r\dot{\theta})\mathbf{u}_\theta$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

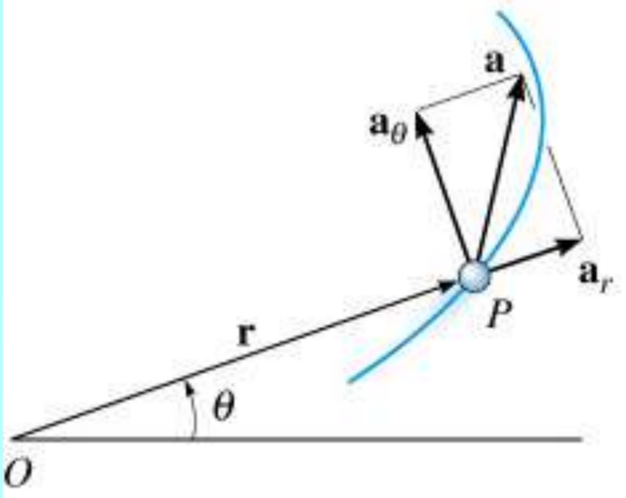
- We again state without proof (see text for details) that

$$\dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r$$

and we recall that we have $\dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta$

- We thus have

$$\begin{aligned}\mathbf{a} &= \ddot{r}\,\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta \\ &= \ddot{r}\,\mathbf{u}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r\dot{\theta}^2\mathbf{u}_r \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta\end{aligned}$$



Acceleration

(e)

ACCELERATION (continued)

- Thus, the component form for the acceleration vector in polar coordinates is given by

$$\mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

- **Nomenclature:** $\ddot{\theta} = d^2\theta / dt^2$ is known as the **angular acceleration**, and is measured in rad/s^2

- **Magnitude of acceleration**

$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

- **Direction of acceleration:** No special direction (**not** tangent to path in general), but as in our discussion of curvilinear motion in general, must be oriented to “swing” velocity vector towards concave side of path