WMAP2D: RNPL IMPLEMENTATION OF 2+1 WAVE MAP

LARS ANDERSSON, MATTHEW W. CHOPTUIK

1. Equations of Motion

Recapping, we have from [1]:

$$\Box u^a = \epsilon u^a g^{\alpha\beta}\partial_\alpha u^b \partial_\beta u^c h_{bc} \tag{1}$$

$$u^a\Big|_{t=0} = v^a, \qquad \partial_t u^a\Big|_{t=0} = w^a \tag{2}$$

where

- In case $N = S^2$, $h_{ab} = \delta_{ab}$, $\epsilon = -1$. initial data should satisfy $v^a v^b h_{ab} = 1$, $v^a w^b h_{ab} = 0$
- In case $N = H^2$, $(h_{ab}) = \mathrm{diag}(-1, 1, 1)$, $\epsilon = +1$. initial data should satisfy $v^a v^b h_{ab} = -1$, $v^a w^b h_{ab} = 0$.

Consider the $N = S^2$ case. Adopting the usual Cartesian coordinates, (t, x, y), we have three (component) wave fields:

$$u^1 := u^1(x, y, t) \tag{3}$$

$$u^2 := u^2(x, y, t) \tag{4}$$

$$u^3 := u^3(x, y, t) \tag{5}$$

which must satisfy

$$(u_1)^2 + (u_2)^2 + (u_3)^2 = 1. \tag{6}$$

For the purposes of finite-differencing, it is convenient to re-write (1) in first-order-in-time form. We thus introduce the "conjugate" variables π^a

$$\pi^a := u^a{}_{,t} \tag{7}$$

We now define an aciliary quantity, $s(x, y, t)$ (the "source function"), via

$$s(x, y, t) := g^{\alpha\beta}\partial_\alpha u^b \partial_\beta u^c h_{bc} \tag{8}$$

Given that

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1) \tag{9}$$

$$h_{ab} = \mathrm{diag}(1, 1, 1) \tag{10}$$

we have

$$s = -(\pi^1)^2 - (\pi^2)^2 - (\pi^3)^2 + (u^1{}_{,x})^2 + (u^2{}_{,x})^2 + (u^3{}_{,x})^2 + (u^1{}_{,y})^2 + (u^2{}_{,y})^2 + (u^3{}_{,y})^2 \tag{11}$$

With the above choices and definitions, (1) now becomes

$$-u^a{}_{,tt} + u^a{}_{,xx} + u^a{}_{,yy} = \epsilon s u^a \tag{12}$$

Date:

which we then rewrite as the system of 6 PDEs for the field variables (u^a, π^a)

$$u^a{}_{,t} = \pi^a \tag{13}$$

$$\pi^a{}_{,t} = u^a{}_{,xx} + u^a{}_{,yy} - \epsilon s u^a \tag{14}$$

$$s = -(\pi^1)^2 - (\pi^2)^2 - (\pi^3)^2 + (u^1{}_{,x})^2 + (u^2{}_{,x})^2 + (u^3{}_{,x})^2 + (u^1{}_{,y})^2 + (u^2{}_{,y})^2 + (u^3{}_{,y})^2 \tag{15}$$

2. Implementation

References

[1] Andersson, L; unpublished notes