THE 3+1 EQUATIONS

- HAVING DERIVED THE 3+1 EQNS IN COORDINATE-FREE FORM

(E1) $$\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab}$$

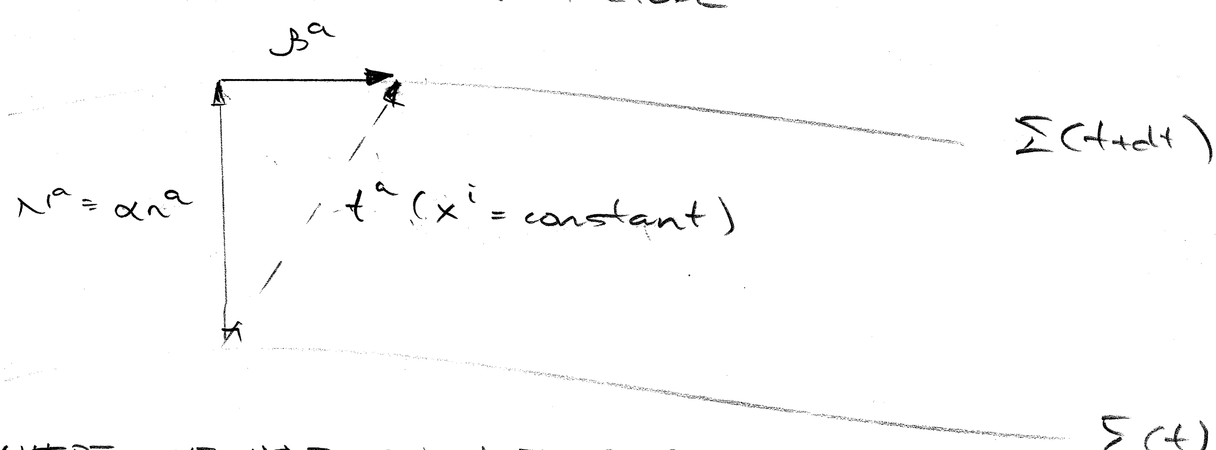
$$= -2\alpha \gamma_{ac} K^c{}_b + \mathcal{L}_\beta \gamma_{ab}$$

(E2) $$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha$$

$$+ \alpha \left(R^a{}_b + K K^a{}_b + 8\pi \left(\tfrac{1}{2} \delta^a{}_b (S-\rho) - S^a{}_b \right) \right)$$

WE NOW WANT TO EXPRESS THE EQUATIONS AS TENSOR-COMPONENT EQUATIONS WITH RESPECT TO OUR 3+1 COORDINATE SYSTEM / COORDINATE BASIS

- RECALL THE BASIC 3+1 PICTURE



WHERE WE HAVE NOW LABELLED THE VARIOUS VECTORS USING COORDINATE-FREE NOTATION

- ALSO RECALL THAT IF $S_{\mu_1 \ldots \mu_\ell}$ ARE THE COMPONENTS of A 4-TENSOR, THEN FOR ANY FOLIATION, $S_{i_1 \ldots i_\ell}$ ARE THE COMPONENTS of A 3-TENSOR

• FINALLY, RECALL THE 3+1 DECOMPOSITION OF ${}^{(4)}g_{\mu\nu}$, n_μ, ${}^{(4)}g^{\mu\nu}$ & n^μ

$${}^{(4)}g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad n_\mu = (-\alpha, 0, 0, 0)$$

$${}^{(4)}g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} \qquad n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$$

NOTE: $\beta_k = {}^{(3)}g_{kj}\beta^j \qquad {}^{(3)}g^{ik}\,{}^{(3)}g_{kj} = \delta^i{}_j$

CLAIM: (1) $\gamma_{ij} = {}^{(3)}g_{ij}$

(2) $\gamma^{ij} = {}^{(3)}g^{ij} \qquad (\to \gamma^{ij}\gamma_{jk} = \delta^i{}_k)$

(3) $\perp^i{}_j = \delta^i{}_j$

$\Rightarrow$ GET VALID COMPONENT EQNS OF MOTION (IN $\{t, x^i\}$ COORDINATE BASIS) VIA $a \to i$, $b \to j$ ETC IN (E1), (E2) AND TREATING ALL QUANTITIES AS 3-TENSORS WHOSE INDICES ARE RAISED/LOWERED VIA 3-METRIC γ^{ij}, γ_{ij}

ALSO $\quad D_i V^j = \partial_i V^j + {}^{(3)}\Gamma^j{}_{ik} V^k$

$$D_i V_j = \partial_i V_j - {}^{(3)}\Gamma^k{}_{ij} V_k \qquad \text{etc}$$

WHERE $\quad {}^{(3)}\Gamma^i{}_{jk} = \gamma^{il}\,{}^{(3)}\Gamma_{ljk}$

$${}^{(3)}\Gamma_{ijk} = \frac{1}{2}\left(\partial_k \gamma_{ij} + \partial_j \gamma_{ik} - \partial_i \gamma_{jk}\right)$$

PROOF of CLAIM: USE $\gamma_{\mu\nu} = {}^{(4)}g_{\mu\nu} + n_\mu n_\nu$ AND ABOVE-QUOTED EXPRESSIONS FOR ${}^{(4)}g_{\mu\nu}$, n_μ ETC.

$$\gamma_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} + n_0 n_0 & {}^{(4)}g_{0j} + n_0 n_j \\ {}^{(4)}g_{0i} + n_0 n_i & {}^{(4)}g_{ij} + n_i n_j \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha^2 + \beta^k \beta_k + \alpha^2 & \beta_j \\ \beta_i & {}^{(4)}g_{ij} \end{bmatrix}$$

$$= \begin{bmatrix} \beta^k \beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix}$$

$$\gamma^{\mu\nu} = \begin{bmatrix} {}^{(4)}g^{00} + n^0 n^0 & {}^{(4)}g^{0j} + n^0 n^j \\ {}^{(4)}g^{0i} + n^0 n^i & {}^{(4)}g^{ij} + n^i n^j \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\alpha^2} + \frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} - \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} - \frac{\beta^j}{\alpha^2} & {}^{(3)}g^{ij} - \frac{\beta^i \beta^j}{\alpha^2} + \frac{\beta^i \beta^j}{\alpha^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & {}^{(3)}g^{ij} \end{bmatrix}$$

→ THUS $\gamma_{ij} = {}^{(3)}g_{ij}$; $\gamma^{ij} = {}^{(3)}g^{ij}$ AS CLAIMED (SO γ_{ij}, γ^{ij} ARE INVERSES)

ALSO: $\perp^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$

$$\perp^i{}_j = \delta^i{}_j + n^i n_j = \delta^i{}_j$$

LIE DERIVATIVES

- RECALL FROM EARLY DISCUSSION OF LIE DERIVATIVE (WALD C.2), AN "EXPANSION" OF $\mathcal{L}_V S^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$

$$\mathcal{L}_V S^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = V^c\left(\nabla_c S^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\right) - \sum_{i=1}^{k} \left(\nabla_c V^{a_i}\right) S^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots b_l} + \sum_{j=1}^{l} \left(\nabla_{b_j} V^c\right) S^{a_1 \ldots a_k}{}_{b_1 \ldots c \ldots b_l}$$

THE ∇_a CAN BE ANY DERIVATIVE OPERATOR (NOT JUST THE METRIC COMPATIBLE ONE) INCLUDING THE ORDINARY DERIVATIVE ∂_a. (NOTE THAT WE HAVE LIE DERIVATIVE TERMS IN BOTH (E1) AND (E2))

CONVERTING (E1) TO 3+1 COMPONENT FORM

(1) $a \to i$, $b \to j$ etc

$$\mathcal{L}_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$= -2\alpha \gamma_{ik} K^k{}_j + \mathcal{L}_\beta \gamma_{ij}$$

(2) CONVERT LIE DERIVATIVES TO EXPRESSIONS INVOLVING ORDINARY DERIVATIVES $_{,k} \equiv \frac{\partial}{\partial x^k} \equiv \partial_k$

$$\mathcal{L}_t (\ldots) = \frac{\partial}{\partial t}(\ldots) = \partial_t (\ldots)$$

$$\mathcal{L}_\beta \gamma_{ij} = \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k$$

(E1') $$\boxed{\partial_t \gamma_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik}\partial_j \beta^k + \gamma_{kj}\partial_i \beta^k}$$

- CAN REWRITE THIS USING D_i (γ_{ij}-COMPATIBLE

EXERCISE DERIVATIVES) AS FOLLOWS

$$\beta^k \partial_k \gamma_{ij} + \gamma_{ik}\partial_j \beta^k + \gamma_{kj}\partial_i \beta^k$$

$$= \beta^k \partial_k \gamma_{ij} + \partial_j(\gamma_{ik}\beta^k) - \beta^k \partial_j \gamma_{ik}$$

$$+ \partial_i(\gamma_{kj}\beta^k) - \beta^k \partial_i \gamma_{kj}$$

$$= \partial_i \beta_j + \partial_j \beta_i - (\partial_j \gamma_{ik} + \partial_i \gamma_{jk} - \partial_k \gamma_{ij})\beta^k$$

$$= \partial_i \beta_j + \partial_j \beta_i - 2\,{}^{(3)}\Gamma_{kij}\beta^k$$

$$= \partial_i \beta_j + \partial_j \beta_i - 2\,{}^{(3)}\Gamma^k{}_{ij}\beta_k$$

$$= D_i \beta_j + D_j \beta_i$$

(E1'') $$\boxed{\partial_t \gamma_{ij} = -2\alpha \gamma_{ik} K^k{}_j + D_i \beta_j + D_j \beta_i}$$

(MTW 21.67)

EVOLUTION EQUATION FOR EXTRINSIC CURVATURE

$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha$$

$$+ \alpha\left(R^a{}_b + K K^a{}_b + 8\pi\left(\tfrac{1}{2}\perp^a{}_b(S-\rho) - S^a{}_b\right)\right)$$

(1) AGAIN, $\mathcal{L}_t K^a{}_b \to \mathcal{L}_t K^i{}_j = \partial_t K^i{}_j$

(2) $\mathcal{L}_\beta K^a{}_b \to \mathcal{L}_\beta K^i{}_j = \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k$

$$\text{(E2')} \quad \boxed{\begin{aligned} \partial_t K^i{}_j &= \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k - D^i D_j \alpha \\ &+ \alpha \left(R^i{}_j + K K^i{}_j + 8\pi \left(\tfrac{1}{2} \delta^i{}_j (S - \rho) - S^i{}_j \right) \right) \end{aligned}}$$

WHERE: $D^i D_j \alpha = \gamma^{ik} D_k D_j \alpha$

$$= \gamma^{ik} D_k (\partial_j \alpha)$$

$$= \gamma^{ik} (\partial_k \partial_j \alpha - \Gamma^l{}_{kj} \partial_l \alpha)$$

$$R^i{}_j = \gamma^{ik} R_{kj} = \gamma^{ik} R_{klj}{}^l$$

$$R_{ijk}{}^l = -2 \partial_{[i} \Gamma^l_{j]k} + 2 \Gamma^m{}_{k[i} \Gamma^l_{j]m}$$

$$\Gamma^i{}_{jk} \equiv {}^{(3)}\Gamma^i{}_{jk} = \tfrac{1}{2} \gamma^{il} (\partial_k \gamma_{lj} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk})$$

$$K = K^i{}_i = \gamma^{ij} K_{ij}$$

$$S_{ij} = T_{ij} \quad ; \quad S^i{}_j = \gamma^{ik} S_{kj}$$

$$\rho = \alpha^2 T_{00}$$