Week 2: Much ado about Hydrodynamics and related problems
Part II: (Truly) General Relativistic Hydrodynamics and beyond

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1 Basic considerations

The description of modern astronomy for strongly gravitating systems or cosmology requires general relativity. In many astrophysical systems, high energy electromagnetic radiation is often emitted in the regions of strong gravitational fields around compact objects. These systems are poorly understood, but data is pouring from all directions! Observationally a number of high energy X-ray, and γ ray telescopes are providing exciting (and challenging) information about a number of sources. Soon, gravitational wave detectors will provide a completely (and often complementary) new way to probe these systems. Luckily to these exciting observational inputs we are now (and even better towards the future) in a position to directly –and from first principles– study these systems to the best of our knowledge and, confrontation with the data will strengthen or modify our understanding of gravity, matter in extreme regimes, etc. An important point here is that as opposed to laboratory experiments, we do not have control on 'nature's experiments' we can not simplify the inputs or the setup, we can however concentrate on relevant scenarios where, we *think* some details are not important. We will discuss some relevant examples later.

2 No EM radiation in vacuum, we need more than nothing

Despite what Hollywood might want us to believe, there can be no radiation in pure vacuum. To do so, we need at least 'hot' matter to produce some type of radiation. Thus, the simplest system we can think off is gravity interacting with the simplest matter form we can think of at a theoretical level. This is the case for a perfect fluid, whose strength energy tensor is given by,

$$T_{ab} = \rho h u_a u_b + p g_{ab} \tag{1}$$

with $u^a u_a = -1$ the four-velocity of the fluid, p its pressure, ρ density and the relativistic specific enthalpy $h \equiv 1 + \epsilon + p/\rho$ with ϵ the specific energy density. This assumption ignores all microphysics and so is just in effective description. For the systems we are interested on, these details are for the most part unimportant. The equations of General Relativistic Hydrodynamics are

$$\nabla_a T^{ab} = 0 \; ; \; \nabla_a J^a = 0 \, . \tag{2}$$

(with $J^a = \rho u^a$). The equations above, together with the normalization condition for u^a, constitute 6 equations for the 7 unknowns (ρ, ϵ, p, u^a); in order to close the system a further equation must be added. This is an equation of state, $p = F(\rho, \epsilon)$. Typical choices are the ideal fluid ($p = (\Gamma - 1)\rho\epsilon$) and a polytrope ($p = \kappa\rho^\Gamma$). Now, the equations defined by the system (2) are truly non-linear, so the issues raised earlier apply here. We stressed that in order to obtain the correct description of shocks and other peculiarities naturally arising in these type of equations, expressing the equations in "conservation form" was useful in exploiting Godunov methods that are designed for truly nonlinear equations.

2.1 A special case, static stars

As an example, let us study –and derive an exercise– the case of a static star in spherical symmetry. Staticity and spherical symmetry implies 3 Killing fields exist and so we can adopt coordinates adapted to express the line element and stress tensor as,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \tag{3}$$

$$T_{ab} = \hat{\rho}(r) u_a u_b + p(r) g_{ab} \tag{4}$$

where we have defined $\hat{\rho}(r) = \rho(r)h(r)$ and $u_a = (-\sqrt{B}, 0, 0, 0)$ Now, Einstein equations –now there is a source!– imply,

$$\frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} = -4\pi(\hat{\rho} - p)A \tag{5}$$

$$-1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = -4\pi(\hat{\rho} - p)r^2 \tag{6}$$

$$-\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} = -4\pi(\rho + 3p)B \tag{7}$$

where a prime $'$ denotes a derivative with respect of r. Now, the equations (2) imply that in our case of stationarity and spherical symmetry,

$$\frac{B'}{B} = -\frac{2p'}{p + \hat{\rho}} \tag{8}$$

PROBLEM 2.1

Prove:

- $u_a = (-\sqrt{B}, 0, 0, 0)$
- Write the components of T_{ab}
- Einstein equations are $G_{ab} = 8\pi T_{ab}$ however they can be written also as $R_{ab} = S_{ab}$ with S_{ab} a symmetric tensor defined in terms of T_{ab}, find S_{ab}.
- Derive equations (5)-(7) and (8).

Now, using equations (5)-(7) obtain an equation which does not involve B at all (ie. it will depend only of A, r and $\hat{\rho}$. Such equation implies

$$\left(\frac{r}{A}\right)' = 1 - 8\pi\hat{\rho}r^2 \tag{9}$$

whose solution is

$$A(r) = \left(1 - 2\frac{M(r)}{r}\right)^{-1} \tag{10}$$

with,

$$M(r) = \int_o^r 4\pi r'^2 \hat{\rho}(r')dr' \tag{11}$$

Since we have now determined A in terms of fluid variables and we already had B in terms of them (equation 8), we can eliminate metric functions from Einstein equations (say 6) and after some simple algebra obtain,

$$-rp' = M\hat{\rho}(r)\left(1 + \frac{p}{\hat{\rho}}\right)\left(1 + \frac{4\pi p}{M}\right)\left(1 - 2\frac{M}{r}\right)^{-1} \tag{12}$$

Thus, equation (12) coupled to the radial derivative of equation (12), together with $M(r = 0) = 0$ and a given value of the central density $\hat{\rho}(r = 0)$ can be used to obtain solutions describing stars in hydrostatic equilibrium in general relativity, provided an equation of state is adopted. An example is a polytrope where $p = \kappa\hat{\rho}^n$. The numerical integration of this problem is easy, it involves ordinary differential equations in one dimension. As an optional assignment, consider trying this out and you will be "building your own star".

3 Beyond GRHydro, magnetic fields and then some

Spectacularly energetic astrophysical events, like AGNs, quazars, blazars, γ ray bursts, typically have jets, where flows of extremely high Lorentz factors are observed which are also collimated in a narrow beam. There is little doubt that magnetic fields play a key role in the collimation and energetics observed though the theoretical understanding of these events is still limited. Important developments are taking place thanks to numerical simulations that are able to concentrate on particular stages of (some of) these events, though there are *a lots* of open issues.

At the theoretical level, to describe these systems one must be able to incorporate the main physics ingredients thought to play a role. General relativity (as compact objects are involved), Hydrodynamics (to describe the dynamics of matter), Electromagnetic fields (to incorporate effects driven by EM fields). To these basic building blocks, ideally one would like to add microphysics and cooling mechanisms as a simple equation of state can not capture the details of the matter behavior and energy carried out by neutrinos and other processes not accounted for by the building blocks above. Depending the particular problem one can argue some of these processes do not significantly affect the dynamics and so a limited description will capture essentially the true behavior of the system. We thus stay within the basic building blocks and describe the underlying theory and relevant issues.

4 The MHD equations in general relativity

We first derive the equations of motion for relativistic MHD and a dynamic spacetime. The equations are written in conservation form as required for High-Resolution Shock-Capturing (HRSC) numerical methods. We then discuss the transformation between conserved and primitive variables.

4.1 Equations of motion

In what follows we combine some notes from different sources. First we deal with the so called em ideal MHD limit. One reference (of many available in the literature) here is [1] To begin, we assume a stress energy tensor of the form

$$T_{ab} = \left[\rho_0 \left(1+\epsilon\right) + P\right] u_a u_b + P g_{ab} + F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd}, \tag{13}$$

where the first few terms describe the fluid and the final two terms the electromagnetic field. The fluid and electromagnetic components are coupled through the relativistic form of Ohm's law:

$$J_a + \left(u_b J^b\right) u_a = \sigma F_{ab} u^b, \tag{14}$$

where J_a is the 4-current. So far what we are describing is General Relativity coupled to both hydro and EM such that $T_{ab} = T^M_{ab} + T^E M_{ab}$ with T^M_{ab} describing the stress energy tensfor of a perfect fluid and $T^E M_{ab}$ the one defined by the electromagnetic fields. At this level, it is often the case that different approximations are employed depending on the system being considered. For instance, if the situation is *inertia* dominated (i.e. the fluid dictates how EM fields lines behave) the ideal MHD approximation arises. If, at the other extreme the dynamics is *electromagnetically dominated* as in the case of plasmas where the inertia of matter is negligible the Force-Free approximation is adopted. We will fist discuss the ideal MHD case and then briefly describe the others.

The ideal MHD approximation is simply the statement that the fluid has perfect conductivity, *i.e.*, $\sigma \to \infty$. Equivalently, this can be expressed as

$$F_{ab}\, u^b = 0, \tag{15}$$

which states that the electric field in the frame of the fluid vanishes. This is sometimes referred to as the "freezing-in" condition of the magnetic field; namely, in the frame of the fluid, the magnetic field lines are frozen to the fluid and carried along with it.

With this in mind, a convenient set of substitutions for the electromagnetic variables is to define 4-covariant "electric" and "magnetic" four-vectors

$$e^a = F^{ab} u_b, \qquad b^a = *F^{ab} u_b, \tag{16}$$

where $*F^{ab} \equiv \epsilon^{abcd} F_{cd}/2$ and ϵ^{abcd} is the standard totally antisymmetric Levi-Civita tensor. Note that we can write these as

$$F_{cd} = u_c e_d - u_d e_c - \epsilon_{cdef} u^e b^f, \qquad *F_{cd} = u_c b_d - u_d b_c + \epsilon_{cdef} u^e e^f, \tag{17}$$

where we have the constraints $u_a e^a = 0 = u_a b^a$. All the information in the Maxwell tensor, F_{ab}, is now contained in these two four vectors.

With these substitutions, the electromagnetic part of the stress tensor can be written as

$$T^{\rm EM}_{ab} = u_a u_b \left[e_c e^c + b_c b^c\right] + \frac{1}{2}\, g_{ab} \left[e_c e^c + b_c b^c\right] - e_a e_b - b_a b_b + 2 u_{(a} \epsilon_{b)cde} e^c u^d b^e. \tag{18}$$

In the MHD approximation, the electric four vector is identically zero and the full stress tensor for MHD can be written as

$$T_{ab} = \left[\rho_0 \left(1+\epsilon\right) + P + b_c b^c\right] u_a u_b + \left[P + \frac{1}{2}\, b_c b^c\right] g_{ab} - b_a b_b. \tag{19}$$

The matter equations of motion can now be written in conservation form

$$\nabla_a T^{ab} = 0, \qquad \nabla_a *F^{ab} = 0. \tag{20}$$

To these must be appended the baryon conservation equation $\nabla_a \left(\rho_0 u^a\right) = 0$.

In a general spacetime we decompose these equations in the usual ADM 3+1 split by projecting along and orthogonal to a unit normal vector, n^a, which is orthogonal to a foliation of spatial hypersurfaces. The projection tensor is

$$h_{ab} = g_{ab} + n_a n_b, \tag{21}$$

with g_{ab} the metric on the 4-manifold. The Einstein equations have the usual 3+1 form with both evolution and constraint equations. Because our focus in this paper is developing a robust MHD code, we will emphasize and solve the flat spacetime equations in later sections. However, our approach in deriving the equations in this section is completely general.

Conservative variables are defined in the conventional way

$$
\begin{aligned}
E &= T_{ab}\, n^a n^b, &(22)\\
S_b &= -T_{ac}\, n^a h_b{}^c, &(23)\\
(\perp T)_{cd} &= T_{ab}\, h^a{}_c h^b{}_d. &(24)
\end{aligned}
$$

With respect to the MHD stress tensor, these give

$$
\begin{aligned}
E &= [\rho_0 (1+\epsilon) + P + b_c b^c] (n^a u_a)^2 - \left[P + \frac{1}{2} b_c b^c\right] - (n_a b^a)^2 , &(25)\\
S_b &= -[\rho_0 (1+\epsilon) + P + b_c b^c] (n^a u_a) (\perp u)_b + (n_a b^a) (\perp b)_b , &(26)\\
(\perp T)_{cd} &= [\rho_0 (1+\epsilon) + P + b_c b^c] (\perp u)_c (\perp u)_d + \left[P + \frac{1}{2} b_c b^c\right] h_{cd} - (\perp b)_c (\perp b)_d , &(27)
\end{aligned}
$$

where we have defined

$$
W \equiv -n^a u_a, \qquad v^a \equiv \frac{1}{W} (\perp u)^a , \tag{28}
$$

and $(\perp X)^a \equiv h^a{}_b X^b$ denotes a projection. Note that W is the Lorentz factor between the fluid frame and the fiducial observers moving orthogonally to the spatial hypersurfaces. In addition, v^a is the (purely spatial) coordinate velocity of the fluid. The matter equations are projected along and orthogonal to n^a, and expressed in terms of the conserved variables

$$
\begin{aligned}
0 &= -n^a \partial_a E + KE - \frac{1}{\alpha^2} D_a \left(\alpha^2 S^a\right) + (\perp T)^{ab} K_{ab}, &(29)\\
0 &= h_{bc} \left[-n^a \partial_a S^b + K S^b + 2 S^a K_a{}^b - \frac{1}{\alpha} S^a \partial_a \beta^b - \frac{1}{\alpha} D_a \left(\alpha (\perp T)^{ab}\right) - \frac{\partial^b \alpha}{\alpha} E\right], &(30)\\
0 &= D_a \left(*F^{ab} n_b\right), &(31)\\
0 &= h_{bc} \left[-n^a \partial_a \left(*F^{de} n_d h^b{}_e\right) + *F^{db} n_d K + \frac{1}{\alpha} D_a \left(\alpha (\perp *F)^{ab}\right) - \frac{1}{\alpha} *F^{da} n_d \partial_a \beta^b\right], &(32)\\
0 &= \frac{1}{\alpha} n^a \partial_a (\alpha D) + \frac{1}{\alpha} D_a (\alpha D v^a) - KW, &(33)
\end{aligned}
$$

where α and β^b are the ADM (3+1) lapse and shift, K_{ab} is the extrinsic curvature, and D_a is the covariant derivative compatible with h_{ab}. These equations, in order, are the energy equation, the Euler equation, the no monopole constraint, the induction (or Faraday) equation and the baryon conservation equation.

It is advantageous to use the standard magnetic field as the evolution variable, rather than the magnetic four vector b^a. This amounts to working in the frame of the fiducial observers moving along n^a instead of in the fluid frame. The electric and magnetic fields in this frame are then

$$
E_a = h_a{}^b F_{bc} n^c, \qquad B_a = \frac{1}{2} \epsilon_{abc} F^{bc}. \tag{34}
$$

where $\epsilon_{abc} \equiv n^d \epsilon_{dabc}$. The ideal MHD approximation then becomes a relation giving the electric field in terms of the magnetic field in the frame of the orthogonally moving observers:

$$E_a = \frac{1}{n_d u^d} \epsilon_{abc} u^b B^c. \tag{35}$$

In practice, two modifications are made to the MHD equations in order to solve them. First, we evolve the quantity $\tau = E - D$ instead of E alone. This is often done to have an energy quantity that reduces to the Newtonian value in the nonrelativistic limit. Secondly, the source term in the induction equation can be eliminated by combining that equation with the no-monopole constraint. The final form for our matter equations thus becomes

$$\partial_t \left(\sqrt{h}\, \tau\right) + \partial_i \left[\sqrt{-g}\left(S^i - \frac{\beta^i}{\alpha}\tau - v^i D\right)\right] = \sqrt{-g}\left[(\perp T)^{ab} K_{ab} - \frac{1}{\alpha} S^a \partial_a \alpha\right], \tag{36}$$

$$\partial_t \left(\sqrt{h}\, S_b\right) + \partial_i \left[\sqrt{-g}\left((\perp T)^i{}_b - \frac{\beta^i}{\alpha} S_b\right)\right] = \sqrt{-g}\left[{}^3\Gamma^i_{ab} (\perp T)^a{}_i + \frac{1}{\alpha} S_a \partial_b \beta^a - \frac{1}{\alpha}\partial_b \alpha\, E\right], \tag{37}$$

$$-\frac{1}{\sqrt{h}} \partial_i \left(\sqrt{h}\, B^i\right) = 0, \tag{38}$$

$$\partial_t \left(\sqrt{h}\, B^b\right) + \partial_i \left[\sqrt{-g}\left(B^b \left(v^i - \frac{\beta^i}{\alpha}\right) - B^i \left(v^b - \frac{\beta^b}{\alpha}\right)\right)\right] = 0, \tag{39}$$

$$\partial_t \left(\sqrt{h}\, D\right) + \partial_i \left[\sqrt{-g}\, D \left(v^i - \frac{\beta^i}{\alpha}\right)\right] = 0. \tag{40}$$

4.2 Primitive and conserved variables

The evolution equations give the time dependence of the conserved variables, $\boldsymbol{u} = (D, S_i, \tau, B_j)^T$, but they also depend on the primitive variables $\boldsymbol{w} = (\rho_0, v_i, P, b_j)^T$. As discussed in this section, for relativistic fluids the transformation from conserved to primitive variables is transcendental. The ability to solve for physical values of the primitive variables under a wide variety of conditions is an important and challenging part of writing a relativistic fluid code.

The conserved variables are

$$D = W\rho_0, \tag{41}$$

$$S_b = (h + b_c b^c)\, W^2\, v_b + (n_a b^a)\,(\perp b)_b\,, \tag{42}$$

$$\tau = (h + b_c b^c)\, W^2 - P - \frac{1}{2} b_c b^c - (n_a b^a)^2 - W\rho_0, \tag{43}$$

$$B^a = -W b^a - u^a \cdot (n^c b_c)\,, \tag{44}$$

where the fluid enthalpy is $h = \rho_0(1 + \epsilon) + P$. To obtain the inverse transformation, we reduce the problem to the solution for the roots of a single nonlinear function. The method is as follows.

We eliminate the magnetic four vector, b^i, from the above equation using

$$b^a = -\frac{1}{W}\left[B^a + u^a \cdot (\perp u)^b B_b\right]. \tag{45}$$

On replacing this, we get

$$\begin{aligned} D &= W\rho_0, & (46)\\ S_i &= \left(hW^2 + B^2\right) v_i - \left(B^j v_j\right) B_i, & (47)\\ \tau &= hW^2 + B^2 - P - \frac{1}{2}\left[\left(B^i v_i\right)^2 + \frac{B^2}{W^2}\right] - W\rho_0, & (48) \end{aligned}$$

where $B^2 \equiv B_i B^i$, $v^2 \equiv v_i v^i$, and the indices are raised and lowered by the spatial metric h_{ij}. The spatial norm of v^i can be expressed in terms of the Lorentz factor

$$W^2 = \frac{1}{1 - v^i v_i}. \tag{49}$$

Density and pressure, two primitive variables, can be expressed as

$$\rho_0 = D\,\frac{1}{W} = D\,\sqrt{1 - v^2}, \qquad P = (h - \rho_0)\,\frac{\Gamma - 1}{\Gamma}. \tag{50}$$

Note that we assume in this section a Γ-law equation of state.

It now remains to find v^i (or W) and h from our knowledge of D, S_i, τ and B_i. We contract B^i with S_i

$$S_i B^i = hW^2\,(B^i v_i), \tag{51}$$

and use this to eliminate $B^i v_i$ in the expressions above for τ and S_i. From $S^i S_i$ we derive the expression

$$\begin{aligned} -(hW^2)^2\,W^2\,S_i S^i + (hW^2)^2\,(hW^2 + B^2)^2\,(W^2 - 1)& \\ -W^2(2hW^2 + B^2)\left(S^i B_i\right)^2 = 0.& \end{aligned} \tag{52}$$

This can be solved for W^2 in terms of conservative variables and the quantity $x \equiv hW^2$:

$$W^2 = \left[1 - \frac{(2x + B^2)(B^j S_j)^2 + x^2(S^j S_j)}{x^2(x + B^2)^2}\right]^{-1}. \tag{53}$$

Finally, we substitute (53) into the equation for τ (which comes about on using our above expressions for the density and pressure):

$$\left[x\left(1 - \frac{\Gamma - 1}{\Gamma}\,\frac{1}{W^2}\right) - D\left(1 - \frac{\Gamma - 1}{\Gamma}\,\frac{1}{W}\right) - \tau + \frac{1}{2}B^2\left(1 + v^2\right)\right] x^2 = \frac{1}{2}\left(B^j S_j\right)^2. \tag{54}$$

The full expression is thus a nonlinear function in x, the roots of which we must calculate. Note that all the coefficients in this expression are conservative variables that on numerical integration of the evolution equations will be known at a given time level. Once x is obtained by solving (54), it is then straightforward to find W^2, v^2, h, ρ_0, P and b^a. Then (54) is solved for x numerically using a combined Newton–Raphson and bisection solver. In practice, a floor is placed on ρ_0 and P, and a typical value for the floor is 10^{-10}. OK... if this is not messy enough, remember that in order to apply Godunov's schemes, we at least need the information of the eigenvalues/eigenvectors. In General Relativist MHD and employing approximate Riemann solvers that only require the maximum characteristic speeds (like HLLE) one is saved from calculating the speeds using that the maximum speed will be that of light. However, higher order solvers or boundary conditions might require a complete knowledge. At present there is not yet a full study of the characteristic structure of the ideal GRMHD equations but work on this front is underway. However, there are a number of situations where the curvature is fixed, or the problem can be treated by Newtonian Mechanics and one "only" implements the ideal (relativistic) MHD equations. For these cases, the characteristic structure is known [4]. For details refer to that paper, but the characteristic speeds are roughly given by,

$$\lambda_e = v^i \; ; \; \lambda_A \; \frac{b^i \pm \sqrt{\mathcal{E}} v^i}{b^t \pm \sqrt{\mathcal{E}} W} \tag{55}$$

with λ_e "entropic" waves, λ_A Alfven waves ($\mathcal{E} = \rho h + b^2$) and another set known as "magnetosonic" waves. One can show that generically all λ's $< c$. In particular, note that the inclusion of magnetic effects implies that perturbations can propagate faster than the fluid speed and so a perturbation at a particular portion of the fluid will affect other areas faster than the "time-of-flight" allowed by the naive fluid motion . This can be seen from a "field lines" point of view where the perturbation is connected to far regions through magnetic field lines. Consequently, magnetic field effects can and do modify strongly the dynamics of particular systems having important observational consequences beyond the "simple" mediation or collimation expected from them. A particular example, for instance, is illustrated in [5].

5 Wait a minute! where did my c go?

Ok... we have gone this far... didn't we loose something along the way? We know Maxwell equations describe propagation at the speed of light, however we just discussed the ideal MHD equations, which involve them and we got speeds quite different from c, what in the world happened?

Well, we took a strange limit, let us get back to the problem and examine it from an agnostic point of view. (These notes are derived from [3]). To simplify the description, let's consider the special relativistic case so that we can forget curvature effects .

The Maxwell equations

The special relativistic Maxwell equations can be written as

$$\partial_b F^{ab} = I^a \, , \tag{56}$$

$$\partial_b {}^*F^{ab} = 0 \, , \tag{57}$$

where F^{ab} and ${}^*F^{ab}$ are the Maxwell and the Faraday tensor respectively and I^a is the electric current 4-vector. A highly-ionized plasma has essentially zero electric and magnetic susceptibilities and the Faraday tensor is then simply the dual of the Maxwell tensor. This tensor provides information about the electric and magnetic fields measured by an observer moving along any timelike vector n^a, namely

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abc} B_c \,. \tag{58}$$

We are considering n^a to be the time-like translational killing vector field in a flat (Minkowski) spacetime, so $n_a = (-1, 0, 0, 0)$ and the Levi-Civita symbol ϵ^{abc} is non-zero only for spatial indices. Note that the electromagnetic fields have no components parallel to n^a (i.e. $E^a \; n_a = 0 = B^a \; n_a$).

By using the decomposition of the Maxwell tensor (58), the equations (56)–(57) can be split into directions which are parallel and orthogonal to n^a to yield the familiar Maxwell equations

$$\nabla \cdot \boldsymbol{E} = q \,, \tag{59}$$
$$\nabla \cdot \boldsymbol{B} = 0 \,, \tag{60}$$
$$\partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\boldsymbol{J} \,, \tag{61}$$
$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0 \,, \tag{62}$$

where we have decomposed also the current vector $I^a = q n^a + J^a$, with q being the charge density, $q n^a$ the convective current and J^a the conduction current satisfying $J^a \; n_a = 0$.

The current conservation equation $\partial_a I^a = 0$ follows from the antisymmetry of the Maxwell tensor and provides the evolution of the charge density q

$$\partial_t q + \nabla \cdot \boldsymbol{J} = 0 \,, \tag{63}$$

which can be obtained also directly by taking the divergence of (61) when the constraints (59)–(60) are satisfied.

The hydrodynamic equations

The evolution of the matter follows from the conservation of the stress-energy tensor

$$\partial_b T^{ab} = 0 \,, \tag{64}$$

and the conservation of baryon number

$$\partial_a (\rho u^a) = 0 \,, \tag{65}$$

where ρ is the rest-mass density (as measured in the rest frame of the fluid) and u^a is the fluid 4-velocity. The stress-energy tensor T^{ab} describing a perfect fluid minimally coupled to an electromagnetic field is given by the superposition

$$T_{ab} = T_{ab}^{\rm fluid} + T_{ab}^{\rm em} \,, \tag{66}$$

where

$$T^{ab}_{\rm em} \equiv F^{ac}F^{b}_{c} - \frac{1}{4}(F^{cd}F_{cd})g^{ab}\,, \tag{67}$$

$$T^{ab}_{\rm fluid} \equiv hu^a u^b + p\; g^{ab}\,. \tag{68}$$

Here $h \equiv \rho(1+\epsilon)+p$ is the enthalpy, with p the pressure and ϵ the specific internal energy.

The conservation law (64) can be split into directions parallel and orthogonal to n^a to yield the familiar energy and momentum conservation laws

$$\partial_t \tau + \nabla\cdot \boldsymbol{F}_\tau = 0\,, \tag{69}$$

$$\partial_t \boldsymbol{S} + \nabla\cdot \mathbf{F}_{\boldsymbol{S}} = 0\,, \tag{70}$$

where we have introduced the conserved quantities $\{\tau, \boldsymbol{S}\}$, which are essentially the energy density $\tau \equiv T_{ab}n^a n^b$and the energy flux density $S_i \equiv T_{ai}n^a$, and whose expressions are given by

$$\tau \equiv \frac{1}{2}(E^2+B^2) + h\; W^2 - p\,, \tag{71}$$

$$\boldsymbol{S} \equiv \boldsymbol{E}\times\boldsymbol{B} + h\; W^2\; \boldsymbol{v}\;. \tag{72}$$

Here $\boldsymbol{v}$ is the velocity measured by the inertial observer and $W \equiv -n_a u^a = 1/\sqrt{1-v^2}$ is the Lorentz factor. The fluxes can then be written as

$$\boldsymbol{F}_\tau \equiv \boldsymbol{E}\times\boldsymbol{B} + h\; W^2\; \boldsymbol{v}\,, \tag{73}$$

$$\mathbf{F}_{\boldsymbol{S}} \equiv -\boldsymbol{E}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{B} + hW^2\boldsymbol{v}\boldsymbol{v} + \left[\frac{1}{2}(E^2+B^2)+p\right]\mathbf{g}\,. \tag{74}$$

Finally, the conservation of the baryon number (65) reduces to the continuity equation written as

$$\partial_t D + \nabla\cdot \boldsymbol{F}_D = 0\,, \tag{75}$$

where we have introduced another conserved quantity $D \equiv \rho W$ and its flux $\boldsymbol{F}_D \equiv \rho W \boldsymbol{v}$.

Ohm's law

As mentioned above, Maxwell equations are coupled to the fluid ones by means of the current 4-vector I^a, whose explicit form will depend in general on the electromagnetic fields and on the local fluid properties. A standard prescription is to consider the current to be proportional to the Lorentz force acting on a charged particle and the electrical resistivity η to be a scalar function. Ohm's law, written in a Lorentz invariant way, then reads

$$I_a + (I^b\; u_b)u_a = \sigma\; F_{ab}\; u^b\,, \tag{76}$$

with $\sigma \equiv 1/\eta$ being the electrical conductivity of the medium. Expressing (76) in terms of the electric and magnetic fields one obtains the familiar form of Ohm's law in a general inertial frame

$$\boldsymbol{J} = \sigma\; W[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - (\boldsymbol{E}\cdot\boldsymbol{v})\boldsymbol{v}] + q\; \boldsymbol{v}\,. \tag{77}$$

Note that the conservation of the electric charge (63) provides the evolution equation for the charge density q (i.e the projection of the 4-current $\boldsymbol{I}$ along the direction $\boldsymbol{n}$), while Ohm's law provides a prescription for the (spatial) conduction current $\boldsymbol{J}$ (i.e. the components of $\boldsymbol{I}$ orthogonal to $\boldsymbol{n}$).

It is important to recall that in deriving expression (77) for Ohm's law we are implicitly assuming that the collision frequency of the constituent particles of our fluid is much larger that the typical oscillation frequency of the plasma. Stated differently, the timescale for the electrons and ions to come into equilibrium is much shorter than any other timescale in the problem, so that no charge separation is possible and the fluid is globally neutral. This assumption is a key aspect of the MHD approximation.

The well-known ideal-MHD limit of Ohm's law can be obtained by requiring the current to be finite even in the limit of infinite conductivity ($\sigma \to \infty$). In this limit Ohm's law (77) then reduces to

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - (\boldsymbol{E} \cdot \boldsymbol{v})\boldsymbol{v} = 0 \,. \tag{78}$$

Projecting this equation along $\boldsymbol{v}$ one finds that the electric field does not have a component along that direction and then from the rest of the equation one recovers the well-known ideal-MHD condition

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} \,, \tag{79}$$

stating that in this limit the electric field is orthogonal to both $\boldsymbol{B}$ and $\boldsymbol{v}$. Such a condition also expresses the fact that in ideal MHD the electric field is not an independent variable since it can be be computed via a simple algebraic relation from the velocity and magnetic vector fields.

Summarizing: the system of equations of the relativistic resistive MHD approximation is given by the constraint equations (59)–(60), evolution equations (61)–(63), (69)–(70) and (75), where the fluxes are given by Eqs. (73)–(74) and the 3-current is given by Ohm's law (77). These equations, together with a equation of state (EOS) for the fluid and a reasonable model for the conductivity, completely describe the system under consideration provided consistent initial and boundary data are defined.

Elementary my dear Watson... the c gets lots in the limit! Different limits of the resistive MHD description

At this point it is useful to point out some properties of the relativistic resistive MHD equations discussed so far, to underline their purely hyperbolic character and to contrast them with those of other forms of the resistive MHD equations which contain a parabolic part instead. To do this within a simple example, we adopt the Newtonian limit of Ohm's law (77),

$$\boldsymbol{J} = \sigma[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \,, \tag{80}$$

where we have neglected terms of order $\mathcal{O}(v^2/c^2)$, obtaining the following potentially stiff equation for the electric field

$$\partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\sigma[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \,. \tag{81}$$

Assuming now a uniform conductivity and taking a time derivative of Eq. (62), we obtain the following hyperbolic equation with relaxation terms (henceforth referred simply as hyperbolic-relaxation equation) for the magnetic field

$$-\frac{1}{\sigma}[\partial_{tt}\boldsymbol{B} - \nabla^2\boldsymbol{B}] = [\partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B})]\,. \tag{82}$$

If the displacement current can be neglected, i.e. $\partial_t\boldsymbol{E} \simeq \partial_{tt}\boldsymbol{B} \simeq 0$, equation (82) reduces to the familiar parabolic equation for the magnetic field

$$\partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) - \frac{1}{\sigma}\nabla^2\boldsymbol{B} = 0\,, \tag{83}$$

where the last term is responsible for the diffusion of the magnetic field. It is important to stress the significant difference in the characteristic structure between equations (82) and (83). Both equations reduce to the same advection equation in the ideal-MHD limit of infinite conductivity ($\sigma \to \infty$) indicating the flux-freezing condition. However, in the opposite limit of infinite resistivity ($\sigma \to 0$) Eq. (83) tends to the (physically incorrect) elliptic Laplace equation $\nabla^2\boldsymbol{B} = 0$ while Eq. (82) reduces to the (physically correct) hyperbolic wave equation for the magnetic field.

6 Geez what a mess! isn't there anything I can do without going nuts?

6.1 The force-free approximation

In the magnetospheres of the neutron stars or black holes the density of the plasma is so low that even moderate magnetic fields stresses will dominate over the pressure gradients. Mathematically, this means that the stress-energy tensor is mainly dominated by the electromagnetic part,

$$T_{\mu\nu} = T^{fluid}_{\mu\nu} + T^{em}_{\mu\nu} \approx T^{em}_{\mu\nu} \tag{84}$$

which conservation law implies that the Lorentz force in negligible

$$\nabla_\nu T^{\mu\nu}_{em} = -F^{\mu\nu} I_\nu \approx 0 \tag{85}$$

The last relation is known as force-free approximation.

The spatial projection of the force-free limit, written in terms of Eulerian observers, is just

$$E^k J_k = 0 \quad , \quad qE^i + \epsilon^{ijk} J_j B_k = 0 \ . \tag{86}$$

By performing the scalar and the vectorial product with $\mathbf{B}$ of the last relation in the previous equation (86), one can obtain that

$$E^i B_i \;=\; 0 \ . \tag{87}$$

$$J^i \;=\; q\frac{\epsilon^{ijk}E_j B_k}{B^2} + J_B\frac{B^i}{B^2} \quad , \tag{88}$$

where $J_B \equiv J^k B_k$ is the component of the current parallel to the magnetic field. The first relation implies that the electric and magnetic fields must be perpendicular while the second defines the current up to the parallel component J_B. By using the Maxwell equations on can compute $\partial_t(E^i B_i)$, which has to vanish due to the constraint (87). This condition imposes a relation for J_B, which can be substituted into (88) in order to complete the equation for the current.

The charge density q can be removed from the evolved variables by using the constraint divergence constraint on the electric field.

Finally, just mention that the characteristic speeds of the Maxwell equations in the force-free limit are given by two Alfven waves and two magnetosonic waves, moving at the speed of light.

PROBLEM 2.2. Derivation of the force-free current in a simple case. Take the system of equations (59-62) and assume for simplicity $q = 0$. The force-free approximation requires $E\dot{B} = 0$, suppose we adjust that to be the case by a judicial initial data.

- Does the evolution equations imply this condition will remain satisfied?
- If it does not, how must one choose the current J so that this is the case?

6.2 A famous example: The Blandford-Znajek model

Motivated by nature of the powering of the AGNs, Blandford and Znajek studied the extraction of rotational energy from a spinning BH by means of the electromagnetic fields [**?**]. This model assumes that the black hole is immersed on the electromagnetic fields produced by a magnetized disk. If the field strength is large enough, the vacuum in unstable to production of electron-positron pairs and thus a force-free region will be established. The rotation of the BH induces a potential difference which will accelerate even a single electron (or positron) to high enough energy to radiate gamma-ray photons by the curvature process. These photons can create an electron-positron pair which will be accelerated again, leading to a cascade. The time-averaged structure of this magnetosphere is described reasonable by the force-free approximation.

We will follow [**?**] to summarize the BZ results. Recall that the Faraday tensor, in terms of the four vector A_a obeys,

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad , \tag{89}$$

For a single black hole, assuming stationarity and axisymmetry ($\partial_\phi \to 0$; $\partial_t \to 0$) *and* that the force-free condition $E^i B_i = 0$ is a good approximation to the situation of interest, one can obtain rather simple expressions for the resulting EM flux of energy.

PROBLEM 2.3, show that

$$E^i B_i = 0 \rightarrow *F^{ab} F_{ab} = 0 \tag{90}$$

Using the expressions above, the vector potentials obeys

$$A_{\phi,\theta} A_{t,r} - A_{t,\phi} A_{\phi,r} = 0 \quad . \tag{91}$$

One can now define a function $\Omega_F(r,\theta)$ such that

$$\Omega_F \equiv -\frac{A_{t,r}}{A_{\phi,r}} = -\frac{A_{t,\theta}}{A_{\phi,\theta}} \quad , \tag{92}$$

which can be interpreted as the "rotation frequency of the electromagnetic field". Since the poloidal field surfaces can be defined by $A_\phi = constant$ (i.e., it is a stream function for the magnetic field), it means that Ω_F and the electrostatic potential A_t are constant along magnetic field lines. Notice that Ω_F can also be written in terms of the Maxwell tensor, namely

$$\Omega_F = \frac{F_{tr}}{F_{r\phi}} = \frac{F_{t\theta}}{F_{\theta\phi}} \quad . \tag{93}$$

Next, recall that the existence of a Killing field ξ^a implies a conserved quantity. defined as $T_{ab}\xi^a$ for each symmetry of the problem. Conservation implies,

$$\partial_b(\xi_a T^{ab}) = 0 \quad . \tag{94}$$

Since we have two killing fields $\xi^a_{(t)} = (1,0,0,0)$ and $\xi^a_{(\phi)} = (0,0,0,1)$ the former defining the electromagnetic energy E and the latter the electromagnetic angular momentum L.

PROBLEM 2.4, show that the radiated electromagnetic energy crossing a spherical surface at a given radius is,

$$\partial_t E = 2\pi \int_0^\pi \sqrt{-g} F_E d\theta \quad , \quad with F_E \equiv -T^r_t \tag{95}$$

PROBLEM 2.5, adopt Kerr-Schild coordinates, and show that the energy flux density at the horizon $r = r_+ = r_H$ is,

$$F_E|_{r=r_H} = 2(B^r)^2 \Omega_F r_H (\Omega_H - \Omega_F) sin^2(\theta). \tag{96}$$

where $r_H = M + \sqrt{M^2 - a^2}$ is the radius of the horizon and $\Omega_H = a/(2Mr_H)$ is the frame dragging frequency at the apparent horizon, usually interpreted as the rotation frequency of the BH.

This result implies that if $0 < \Omega_F < \Omega_H$ and $B^r \neq 0$, then there is an outward directed energy flux at the horizon; rotational energy is being extracted from the black hole due to the magnetic field lines threading the ergosphere and channeling particles through it. The use of Kerr-Schild coordinates allow for direct computations of the flux at the horizon without any special treatment.

Naturally, to get a particular solution one still needs to evaluate B^r and Ω_F, which requires solving the Maxwell equations in the force-free approximation. Nevertheless the qualitative features of this effect will remain unchanged.

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